Carroll Geometry in Gravity, Holography and Strings

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Based on several works with Stefano Baiguera, Troels Harmark,
Dennis Hansen, Jelle Hartong, Niels Obers, Watse Sybesma and Benjamin Søgaard

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Outline

- Why not Lorentzian?
- Newton-Cartan and Carroll geometry
- Carroll expansion of general relativity
- Carroll field theories and flat holography
- Summary and outlook

Why not Lorentzian?

What's wrong with Lorentzian symmetry?
Nothing, but *general relativity is hard!*

Einstein gravity contains Newtonian gravity,

$$g_{00} = -(1 + 2\Phi), \quad v/c \ll 1, \quad \text{weak coupling} \;\; G \ll 1$$

but where is the geometry?

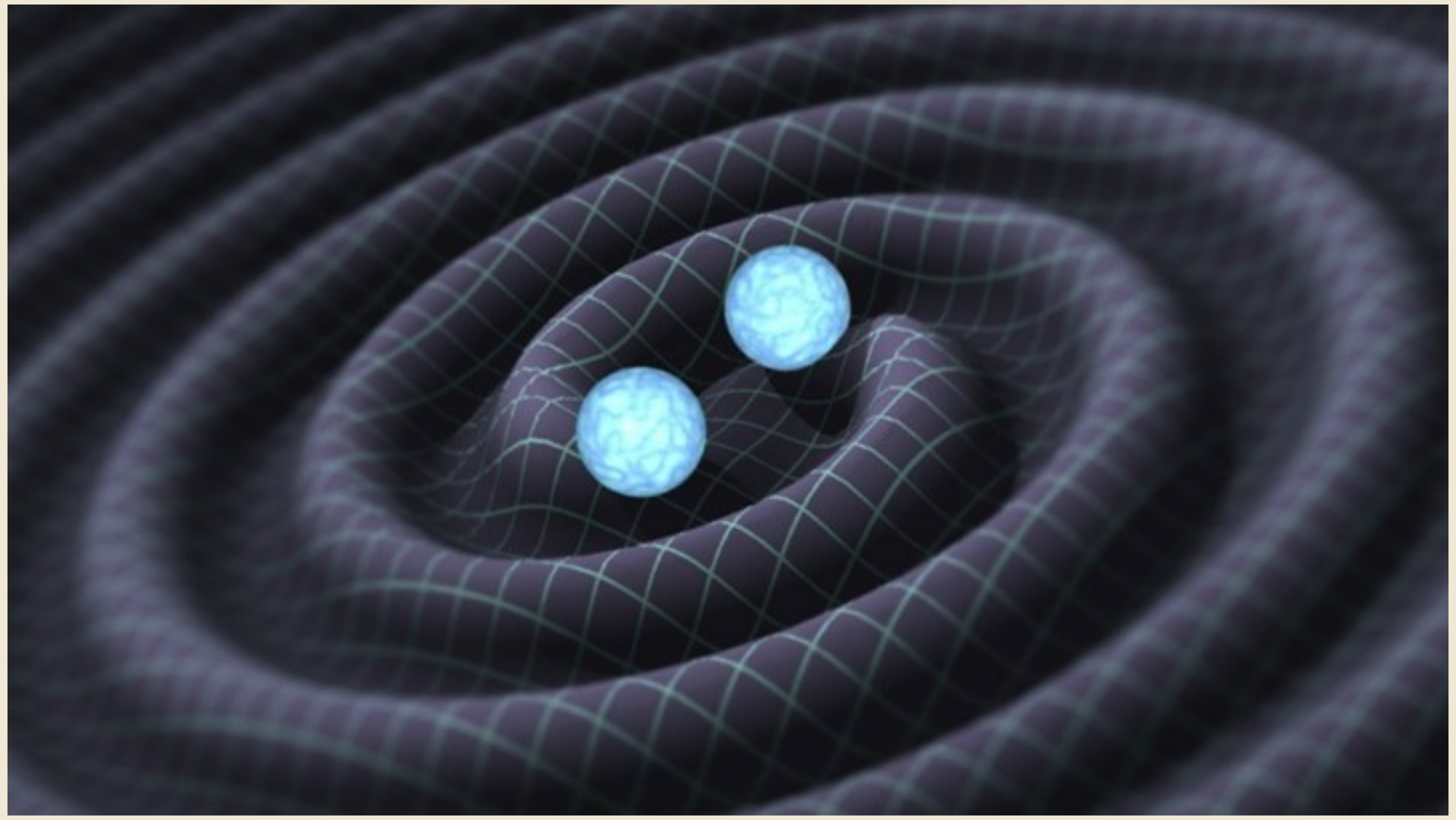
$\Longrightarrow$ Newton-Cartan geometry! [Cartan] [Künzle] [Dautcourt] …

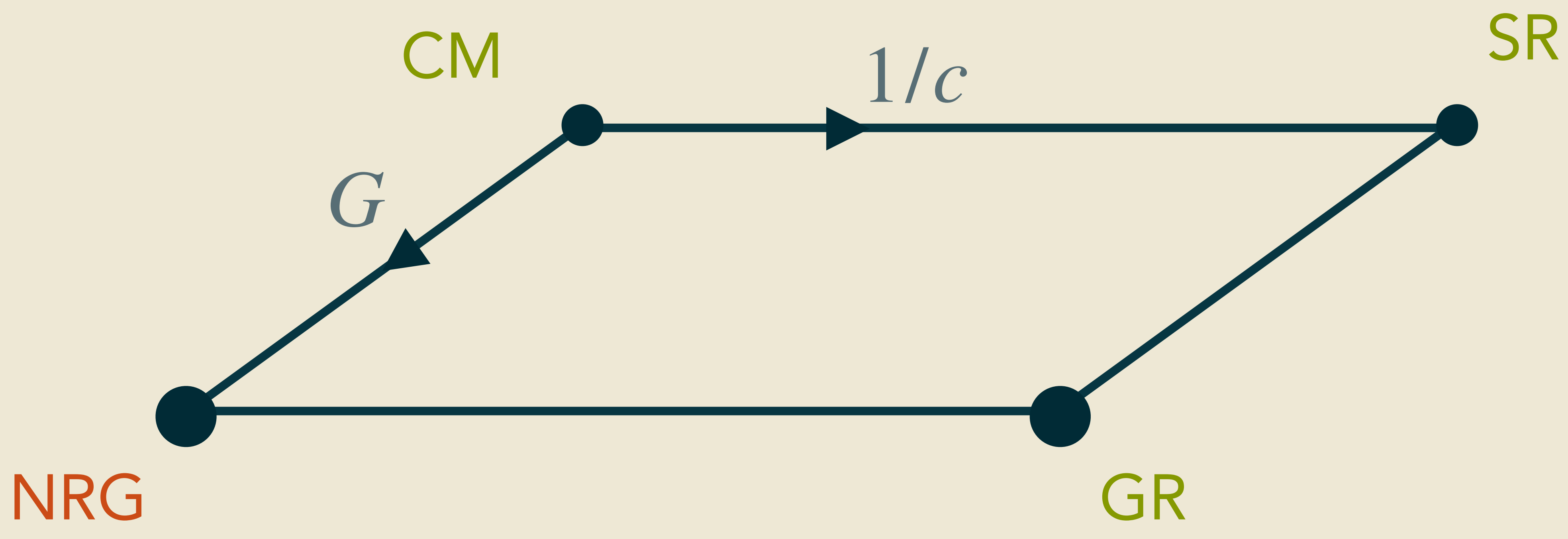
Now understand better [Van den Bleeken] [Hansen, Hartong, Obers]

- how Newton-Cartan geometry arises from Lorentzian
- how Newtonian gravity arises from GR
- weak coupling and low velocity are independent!

Main tool: covariant expansion of geometry in powers of c

see also review [Hartong, Obers, GO]





Why not Lorentzian?

What's wrong with Lorentzian symmetry?
Nothing, but *string theory is also hard!*

Simpler subsector: non-relativistic strings and quantum gravity

[Gomis, Ooguri] [Danielsson, Guijosa, Kruczenski]

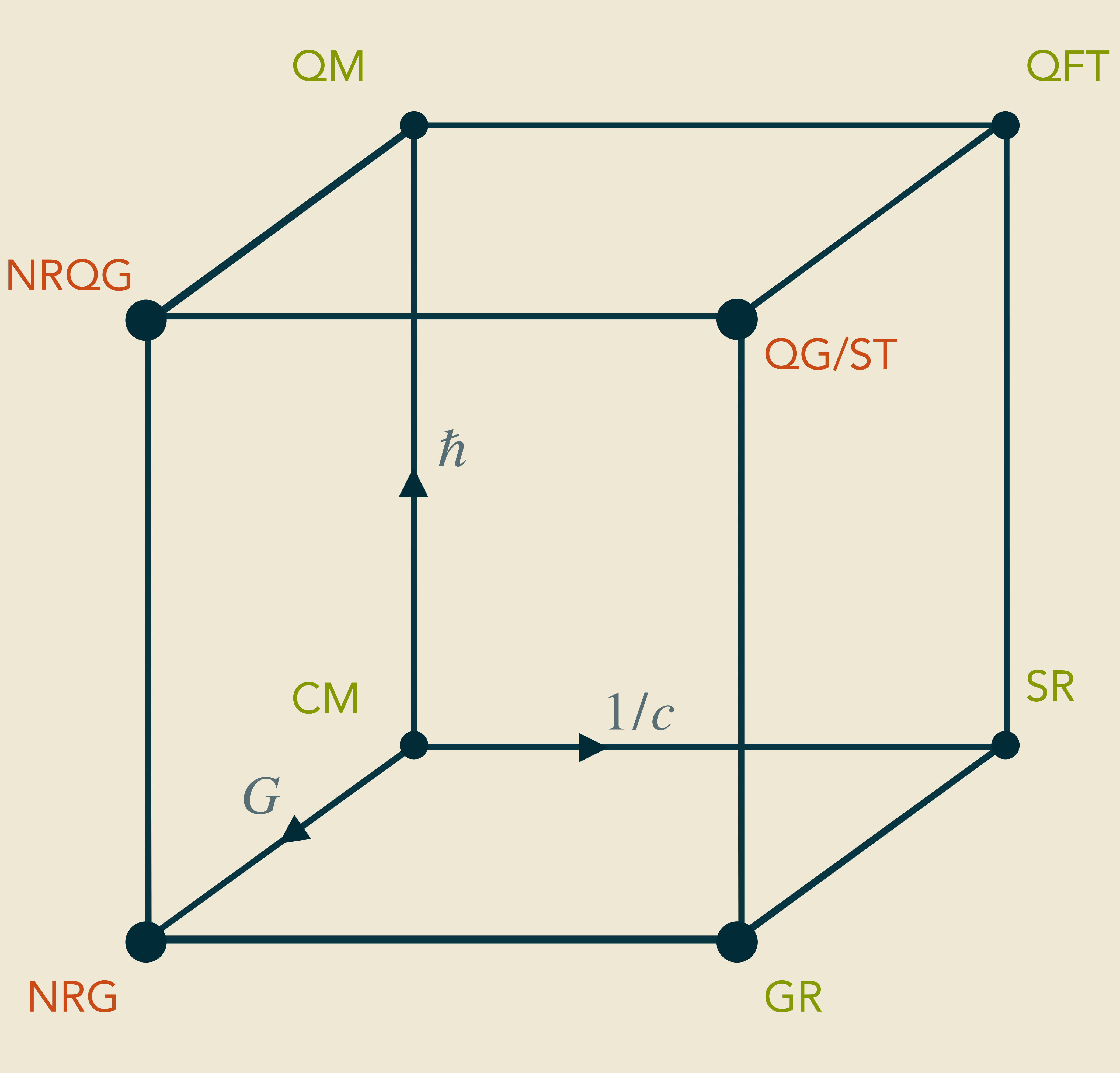
- decoupling limit of string theory
- non-relativistic spectrum
- easier worldsheet theory?

see also review [GO, Yan] or ask Chris Blair!

other non-Lorentzian reviews [Bergshoeff, Gomis, Figueroa-O'Farrill] [Baiguera]

This talk: ultra-local $c \to 0$ Carroll limit

- other tractable limits?
- flat space holography?



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Newton-Cartan and Carroll geometry

compare: Lorentzian geometry

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_2$$

Compatible torsion-free connection $\nabla_\rho g_{\mu\nu} = 0$

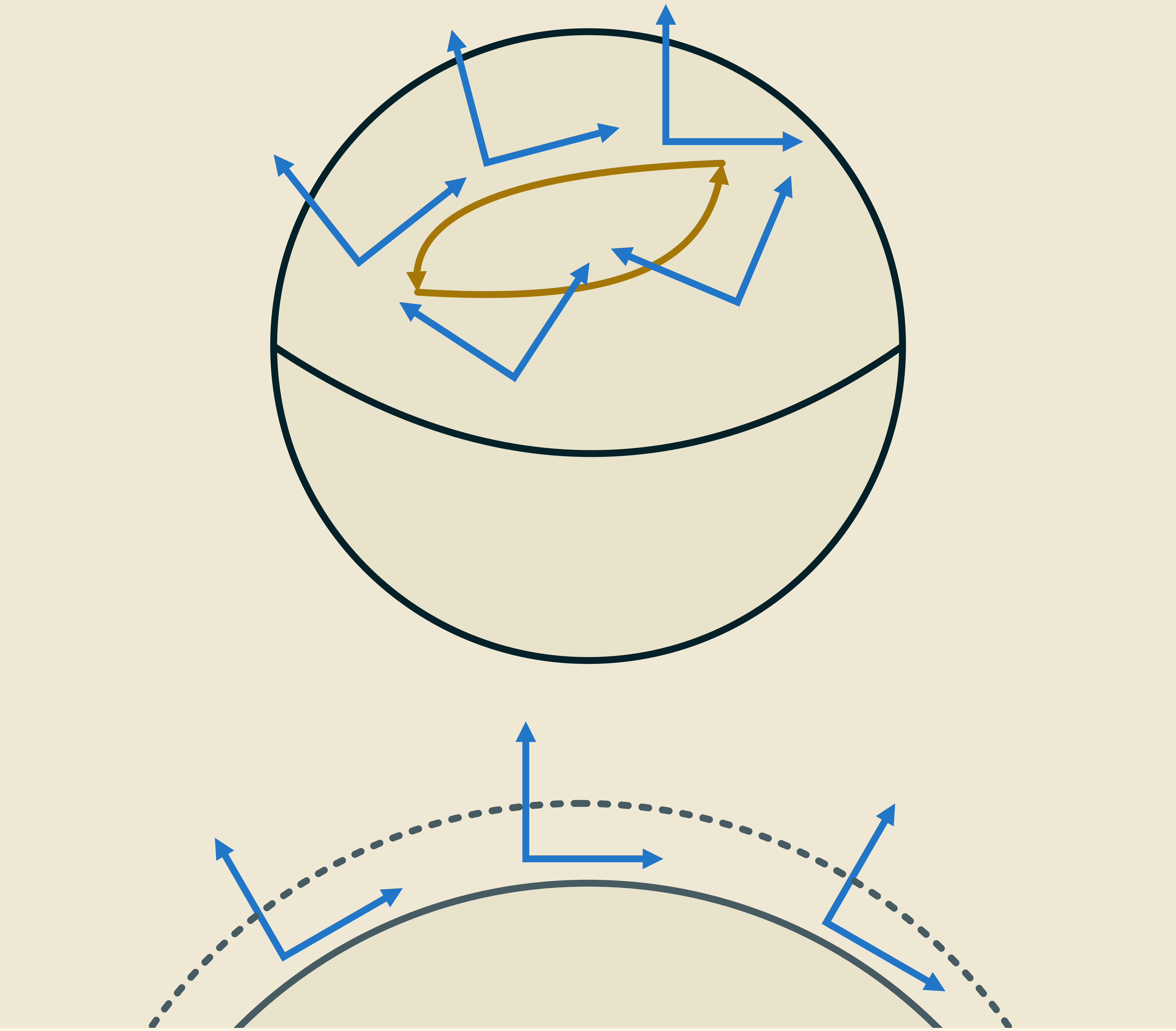
defines curvature $[\nabla_\mu, \nabla_\nu]\, X^\sigma = -R_{\mu\nu\rho}{}^\sigma X^\rho$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{AB} \begin{pmatrix} \sqrt{f} & 0 \\ 0 & 1/\sqrt{f} \end{pmatrix}_\mu{}^A \begin{pmatrix} \sqrt{f} & 0 \\ 0 & 1/\sqrt{f} \end{pmatrix}_\nu{}^B$$

$$= \eta_{AB}\, e_\mu{}^A e_\nu{}^B$$

metric has local Minkowski structure

Mirror this for local Galilean and local Carroll structures



Newton-Cartan geometry

Galilean boost

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

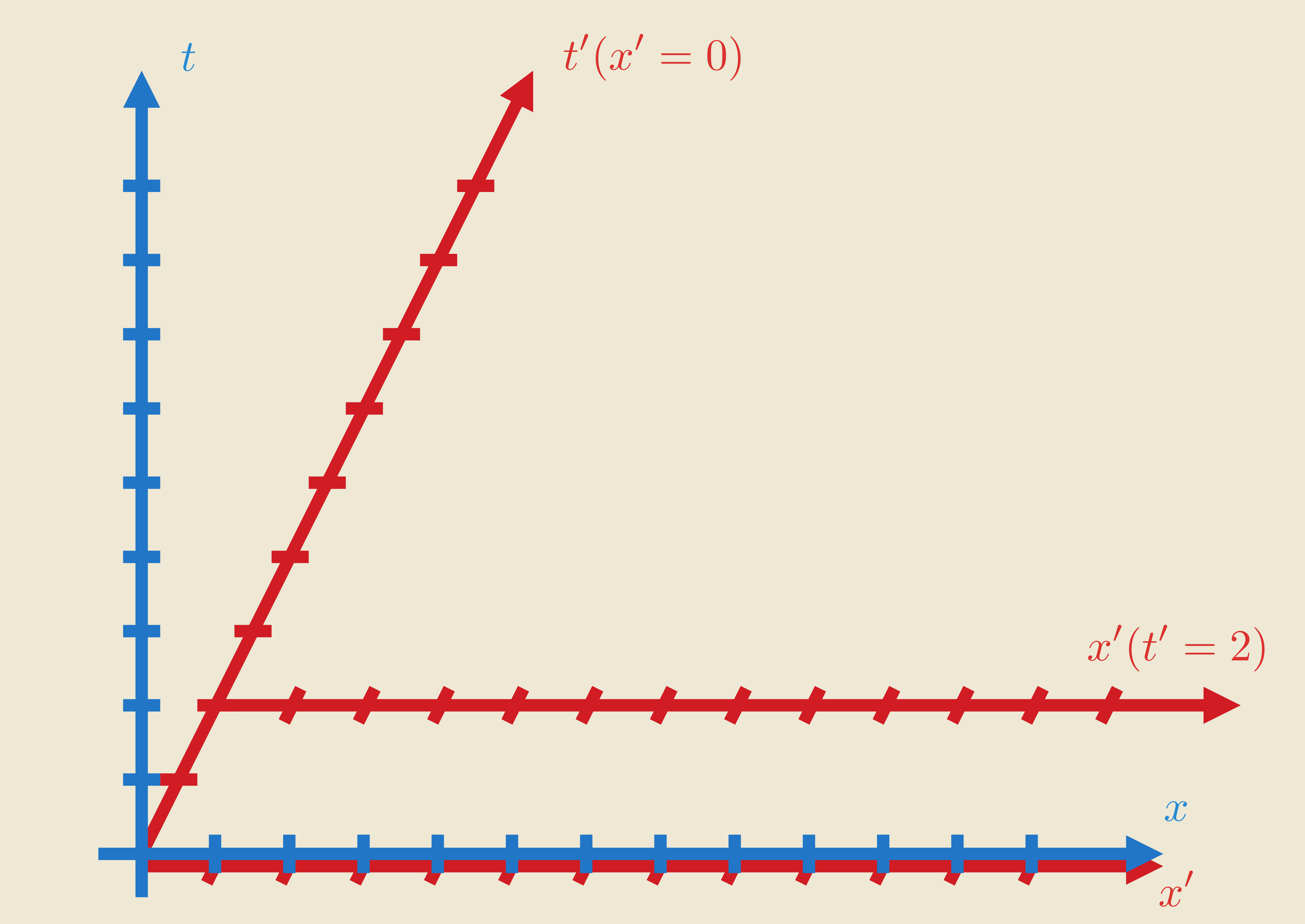
preserves

- time coordinate $\begin{pmatrix} 1 & 0 \end{pmatrix}$
- space direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For curved geometry:

- clock one-form $\tau_\mu dx^\mu \sim \begin{pmatrix} 1 & 0 \end{pmatrix}$
- spatial cometric $h^{\mu\nu}\partial_\mu\partial_\nu \sim$ several $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

known as Newton-Cartan geometry



Newton-Cartan geometry

Newton-Cartan geometry $\tau_\mu(x^\rho)$ and $h^{\mu\nu}(x^\rho)$

Has local Galilean structure!

$$\tau_\mu \sim (1 \quad 0) \quad \text{and} \quad h^{\mu\nu} = \delta^{ab} e^\mu{}_a e^\nu{}_b \sim \text{several} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

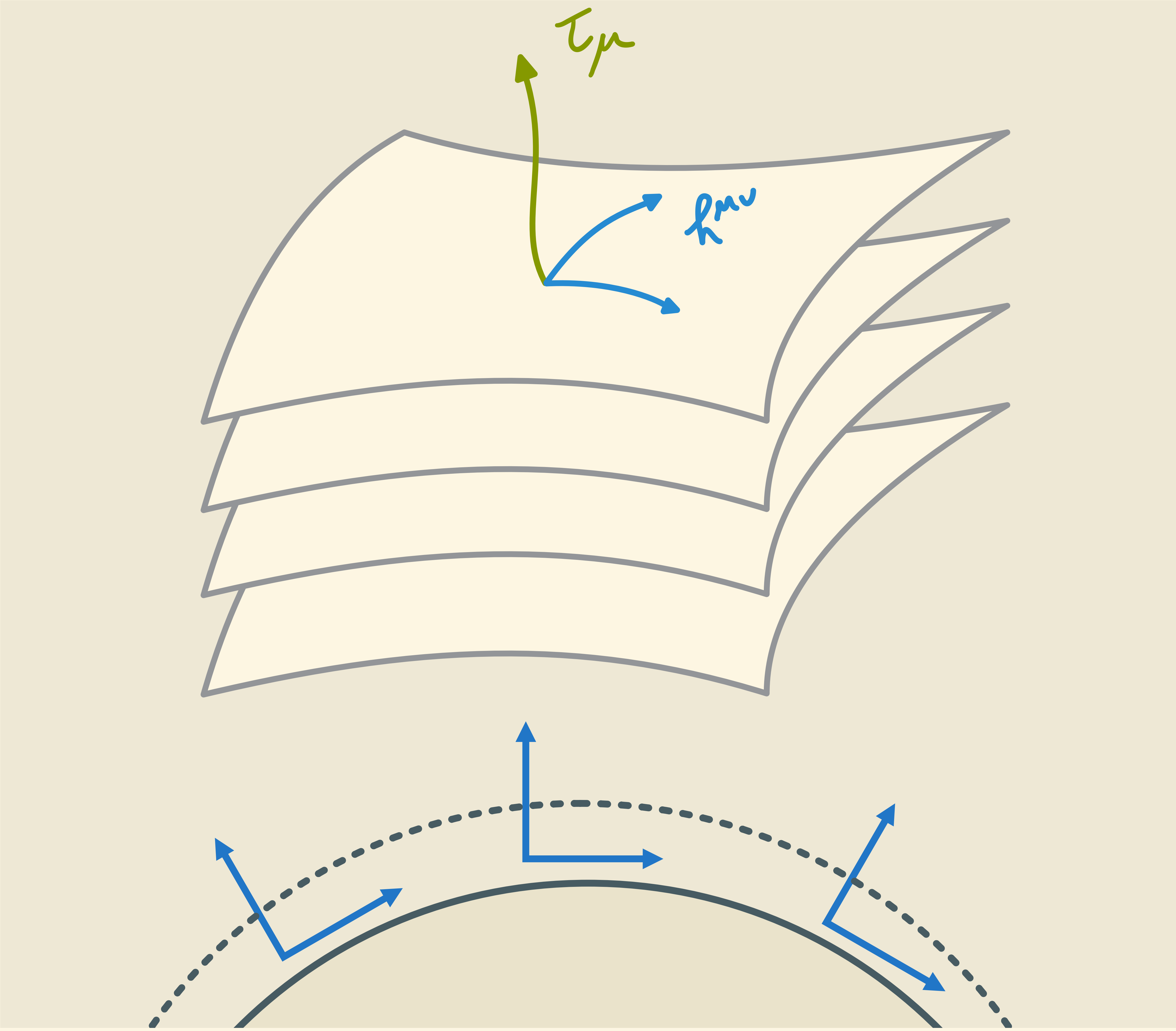
Clock form defines spatial foliation (if $\tau \wedge d\tau = 0$), e.g.

$$\tau_\mu dx^\mu = -\sqrt{1 - \frac{R}{r}} dt \,, \quad h^{\mu\nu}\partial_\mu \partial_\nu = \left(1 - \frac{R}{r}\right) \partial_r^2 + \frac{1}{r^2}\partial_{\Omega_2}$$

Compatible connection $\check{\nabla}_\rho \tau_\mu = 0$ and $\check{\nabla}_\rho h^{\mu\nu} = 0$

curvature $[\check{\nabla}_\mu, \check{\nabla}_\nu] X^\sigma = -\check{R}_{\mu\nu\rho}{}^\sigma X^\rho$

torsion $2\check{\Gamma}^\rho_{[\mu\nu]} = 2\tau^\rho \partial_{[\mu}\tau_{\nu]}$ determined by $d\tau$



Newton-Cartan gravity

Clock one-form $\tau_\mu(x^\rho)$ and spatial cometric $h^{\mu\nu}(x^\rho)$

- get *absolute* time if $d\tau = 0$
  - then $\tau = dt$
  - time t is path-independent
- get time dilation if $d\tau \neq 0$!

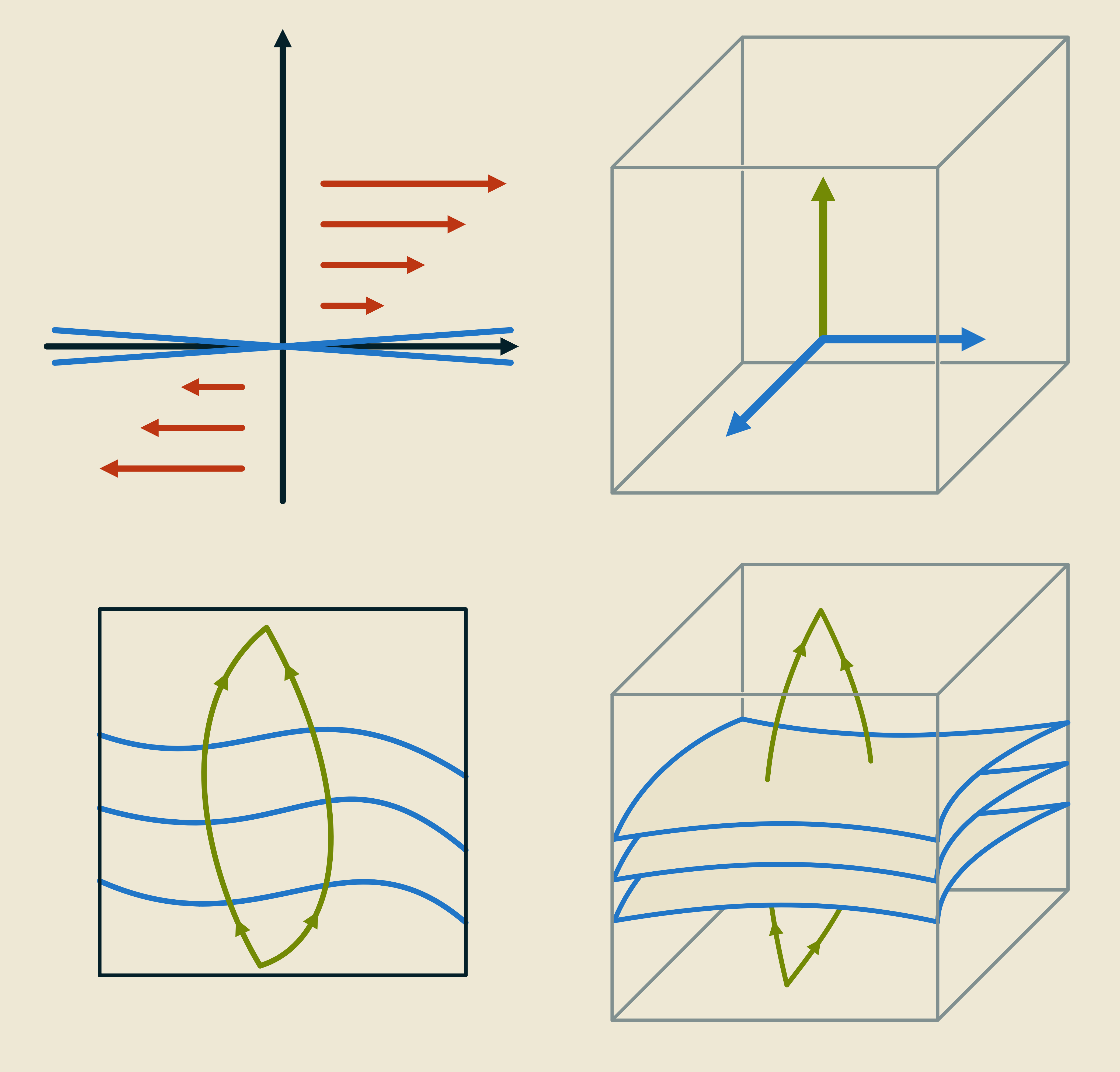
Dynamical gravity from same ingredients:

- Ricci curvature $\check{R}_{\mu\nu\rho}{}^{\sigma}$
- energy-momentum tensor $\check{T}^{\mu}{}_{\nu}$

Modern perspective [Van den Bleeken] [Hansen, Hartong, Obers]

arises from covariant expansion of GR around $c \to \infty$

Leads to 'type II' Newton-Cartan, see review [Hansen, Obers, GO]



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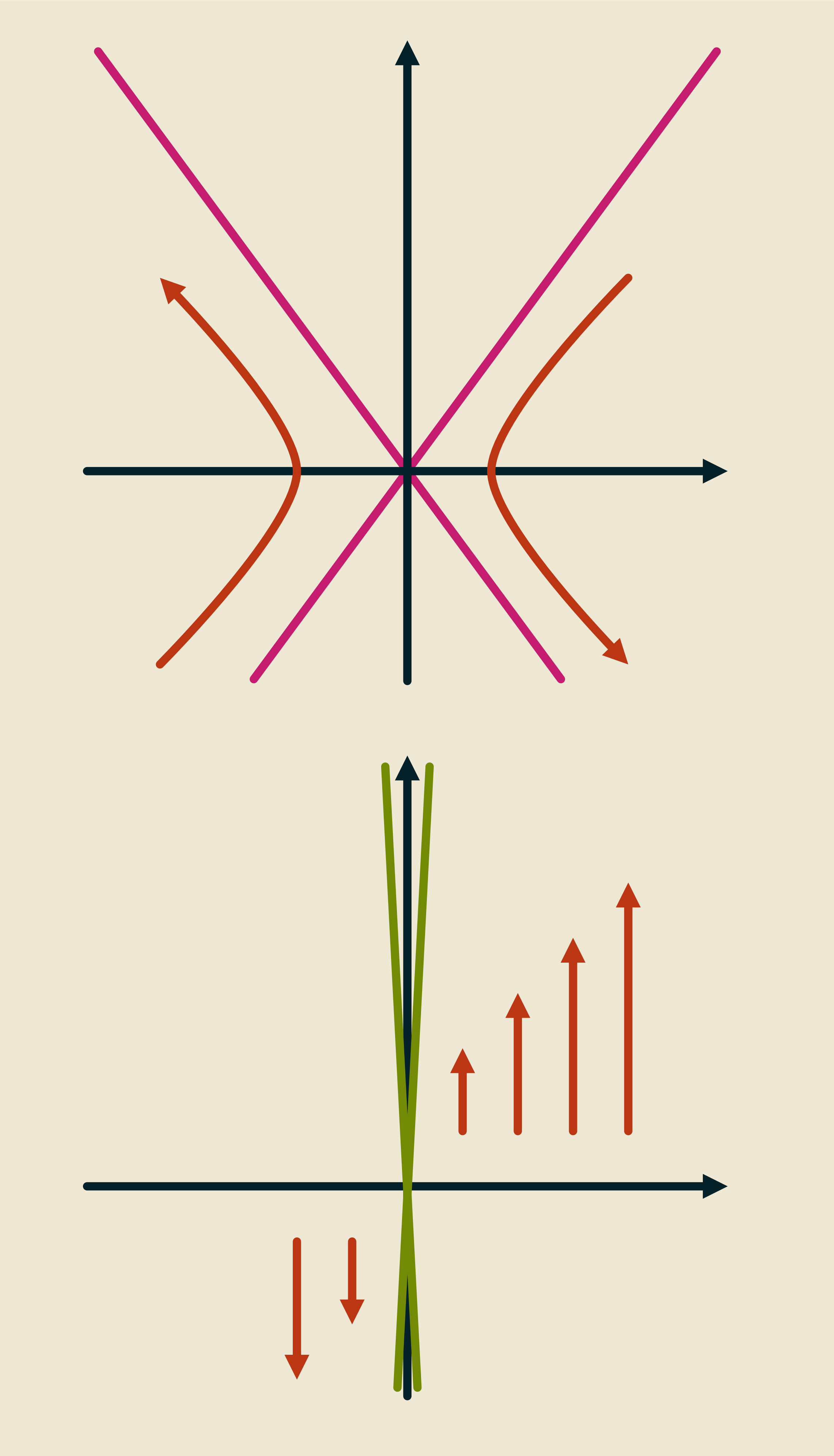
Carroll geometry

Starting from 'relativistic' Lorentz boosts

$$t \to t + \beta x, \qquad x \to x + \beta t$$

get Carroll boosts in ultra-local limit $c \to 0$, [Levy-Leblond] [Sen Gupta]

$$t \to t + \lambda x\,, \quad x \to x \quad \text{and} \quad \partial_t \to \partial_t\,, \quad \partial_x \to \partial_x + \lambda \partial_t$$



Less obviously physical, but

- ultra-local behavior leads to solvable systems
- appears in Lorentzian geometry on null surfaces such as $\mathscr{I}^+$
- BMS asymptotic symmetries are isomorphic to conformal Carroll algebra [Duval, Gibbons, Horvathy, Zhang]

Carroll geometry

Carroll boost

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

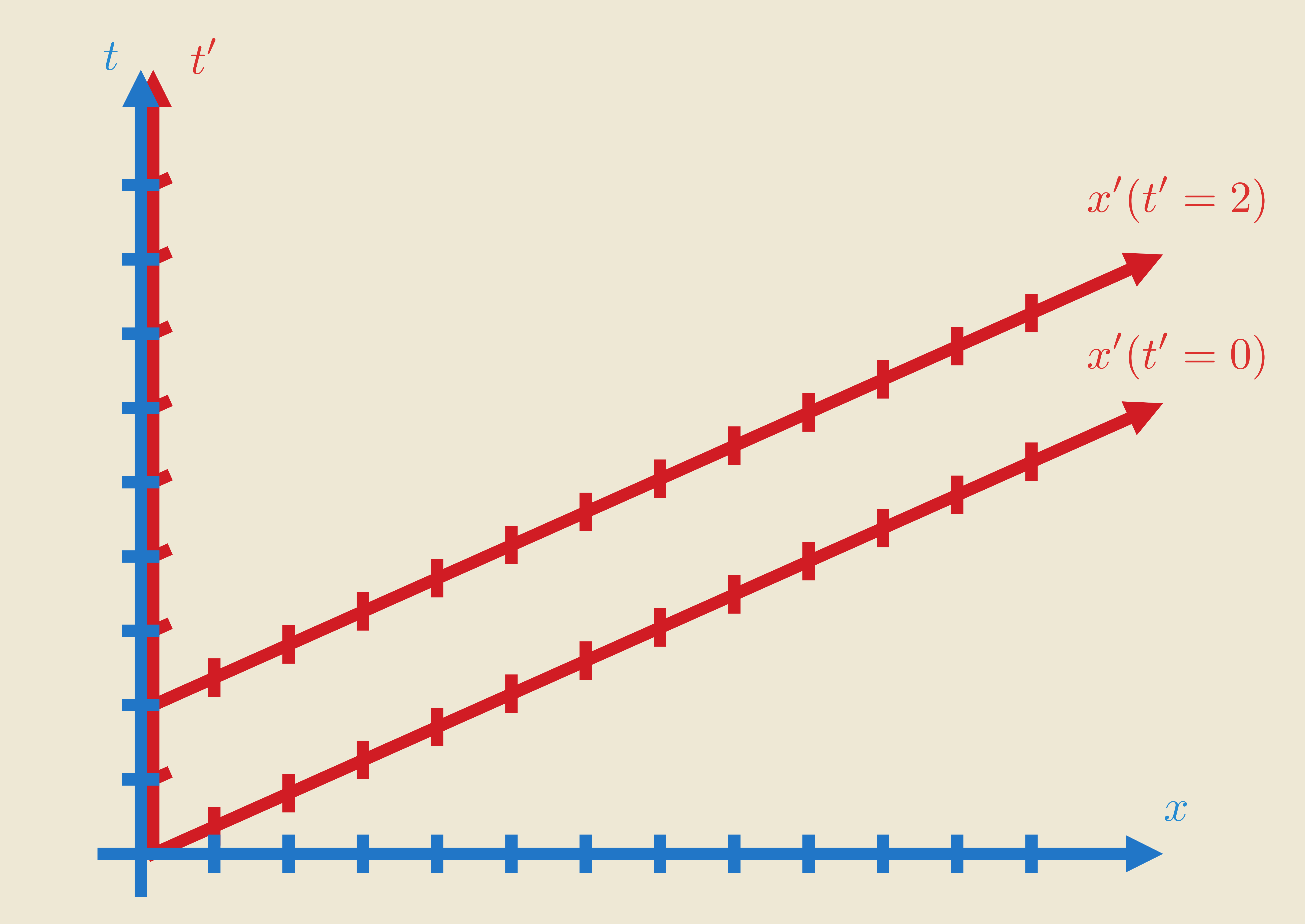
preserves

- time direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- space coordinate $(0 \quad 1)$

For curved geometry:

- time vector field $v^\mu \partial_\mu \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- spatial metric $h_{\mu\nu} dx^\mu dx^\nu \sim$ twice $(0 \quad 1)$

known as Carroll geometry



Carroll geometry

Carroll geometry $v^\mu(x^\rho)$ and $h_{\mu\nu}(x^\rho)$

Has local Carrollian structure!

$$v^\mu \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad h_{\mu\nu} = \delta_{ab} e_\mu{}^a e_\nu{}^b \sim \text{several } (0 \quad 1)$$

Time vector field defines line bundle

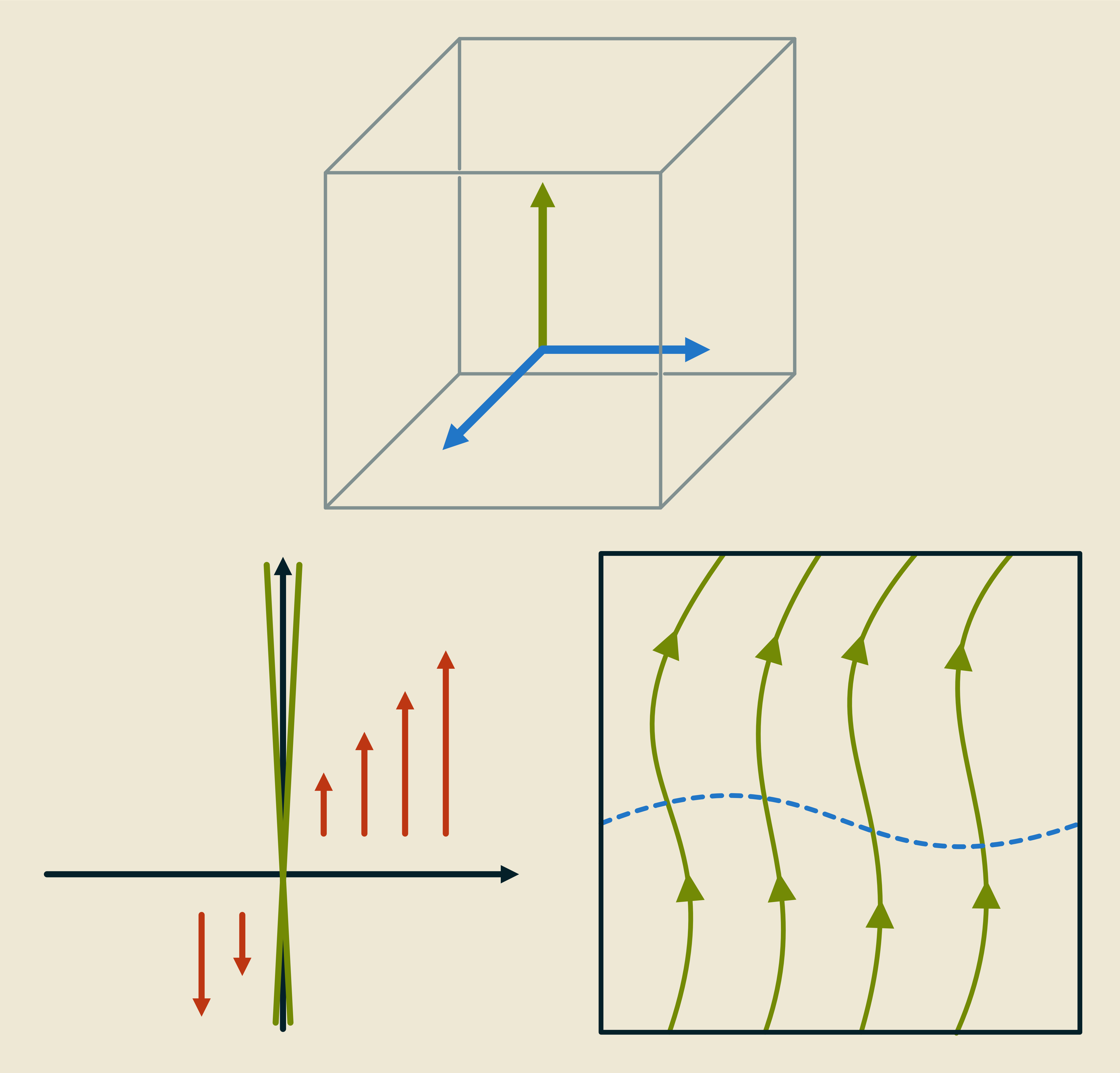
$$v^\mu \partial_\mu = -\partial_t \,, \quad h^{\mu\nu}\partial_\mu \partial_\nu = t^{2p_x} dx^2 + t^{2p_y} dy^2 + t^{2p_z} dz^2$$

Compatible connection $\tilde{\nabla}_\rho v^\mu = 0$ and $\tilde{\nabla}_\rho h_{\mu\nu} = 0$

curvature $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\, X^\sigma = -\tilde{R}_{\mu\nu\rho}{}^\sigma X^\rho$

torsion $2\tilde{\Gamma}^\rho_{[\mu\nu]} = 2h^{\rho\sigma}\tau_{[\mu}K_{\nu]\sigma}$

determined by extrinsic curvature $K_{\mu\nu} = -\frac{1}{2}\mathscr{L}_v h_{\mu\nu}$



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Caroll from Lorentzian

From Lorentzian geometry get Carroll plus corrections by expanding around $c \to 0$

Two-step process [Hansen, Obers, GO, Søgaard]

Rewrite: Choose time vector V^μ and rewrite

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}\,, \qquad g^{\mu\nu} = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}$$

exposes overall factors of c^2 in the metric

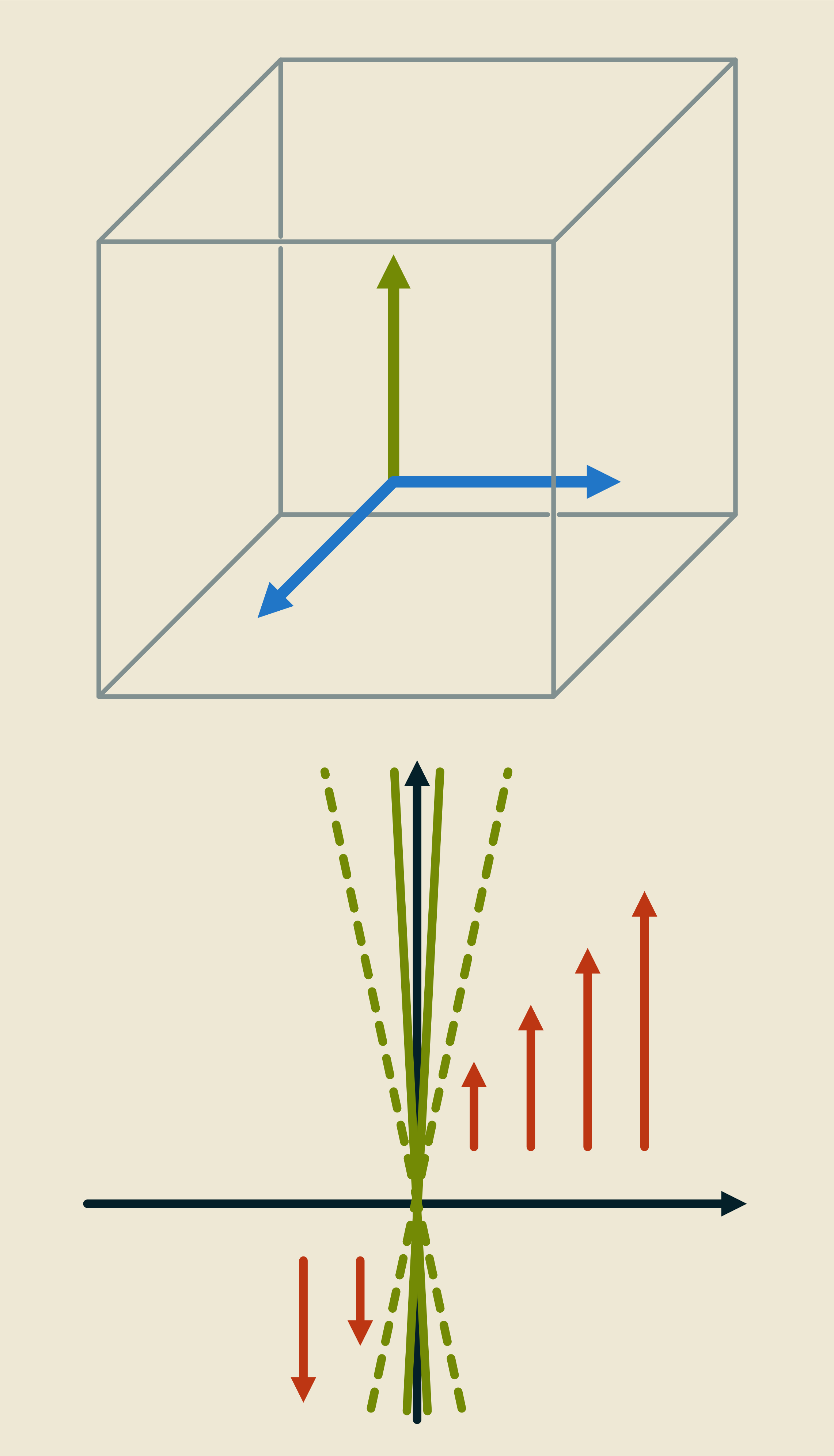
Expand: Carroll geometry at leading order in c^2 expansion

$$T_\mu = \tau_\mu + c^2 m_\mu + \cdots, \qquad V^\mu = v^\mu + \cdots$$

$$\Pi^{\mu\nu} = h^{\mu\nu} + c^2 \Phi^{\mu\nu} + \cdots, \qquad \Pi_{\mu\nu} = h_{\mu\nu} + \cdots$$

local Lorentz boosts → local Carroll boosts + corrections

see also [Bergshoeff, Izquierdo, Ortín, Romano] for algebraic perspective



Caroll from Lorentzian

Carroll connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$ and curvature $\tilde{R}_{\mu\nu\rho}{}^{\sigma}$ from Levi-Civita

Rewrite Levi-Civita with explicit factors of c^2

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{c^2} S_{(-2)}{}^{\rho}{}_{\mu\nu} + \bar{C}^{\rho}_{\mu\nu} + S_{(0)}{}^{\rho}{}_{\mu\nu} + c^2 S_{(2)}{}^{\rho}{}_{\mu\nu}\,,$$

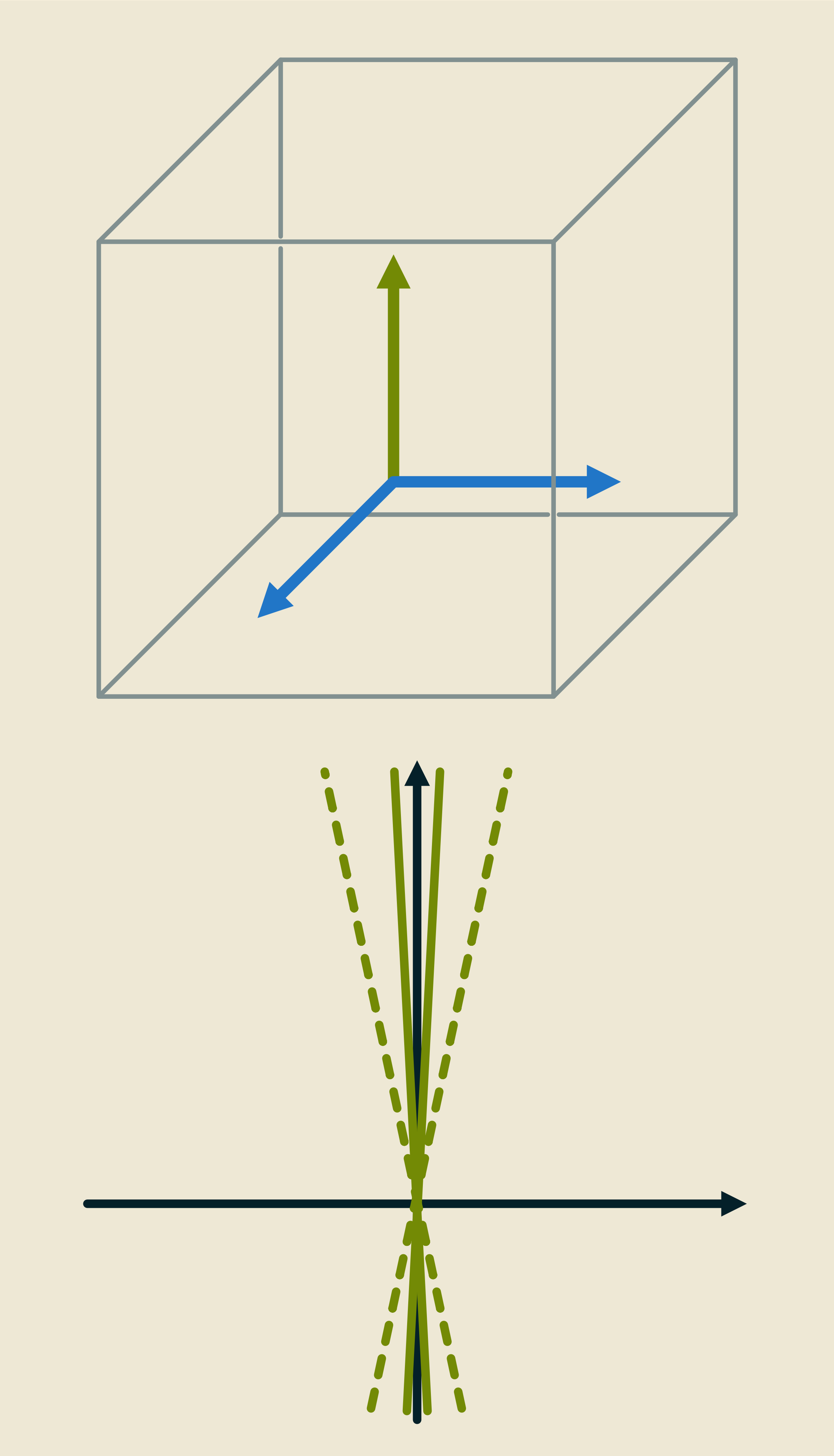
where $S_{(n)}{}^{\rho}{}_{\mu\nu}$ are known tensors. Expand to get $\bar{C}^{\rho}_{\mu\nu} = \check{\Gamma}^{\rho}_{\mu\nu} + \cdots$

Get connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$ so that $\tilde{\nabla}_{\mu} v^{\nu} = 0$ and $\tilde{\nabla}_{\rho} h_{\mu\nu} = 0$,

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = -\, v^{\rho}\partial_{(\mu}\tau_{\nu)} - v^{\rho}\tau_{(\mu}\mathscr{L}_{v}\tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}h_{\nu\sigma} + \partial_{\nu}h_{\sigma\mu} - \partial_{\sigma}h_{\mu\nu}\right) - h^{\rho\sigma}\tau_{\nu}K_{\mu\sigma}$$

Non-zero torsion $\tilde{T}^{\rho}{}_{\mu\nu} = 2h^{\rho\sigma}\tau_{[\mu}K_{\nu]\sigma}$

Rewrite $\sqrt{-g} = cE$ where $E = \det(T_{\mu}, \Pi_{\mu\nu})$ and expand $E = e + \cdots$ where $e = \det(\tau_{\mu}, h_{\mu\nu})$



Caroll from Lorentzian

Rewrite the Einstein-Hilbert action, $\mathscr{K}_{\mu\nu} = -\frac{1}{2}\mathscr{L}_V \Pi_{\mu\nu} = K_{\mu\nu} + \cdots$ is extrinsic curvature

$$S = \frac{c^3}{16\pi G}\int_M R\sqrt{-g}\, d^d x$$
$$\approx \frac{c^2}{16\pi G}\int_M \left[\left(\mathscr{K}^{\mu\nu}\mathscr{K}_{\mu\nu} - \mathscr{K}^2\right) + c^2\,\Pi^{\mu\nu}\bar{R}_{\mu\nu} + c^4\,\Pi^{\mu\rho}\Pi^{\nu\sigma}\partial_{[\mu}T_{\nu]}\partial_{[\rho}T_{\sigma]}\right] E\, d^d x$$

From Lorentzian point of view a strange thing to do!

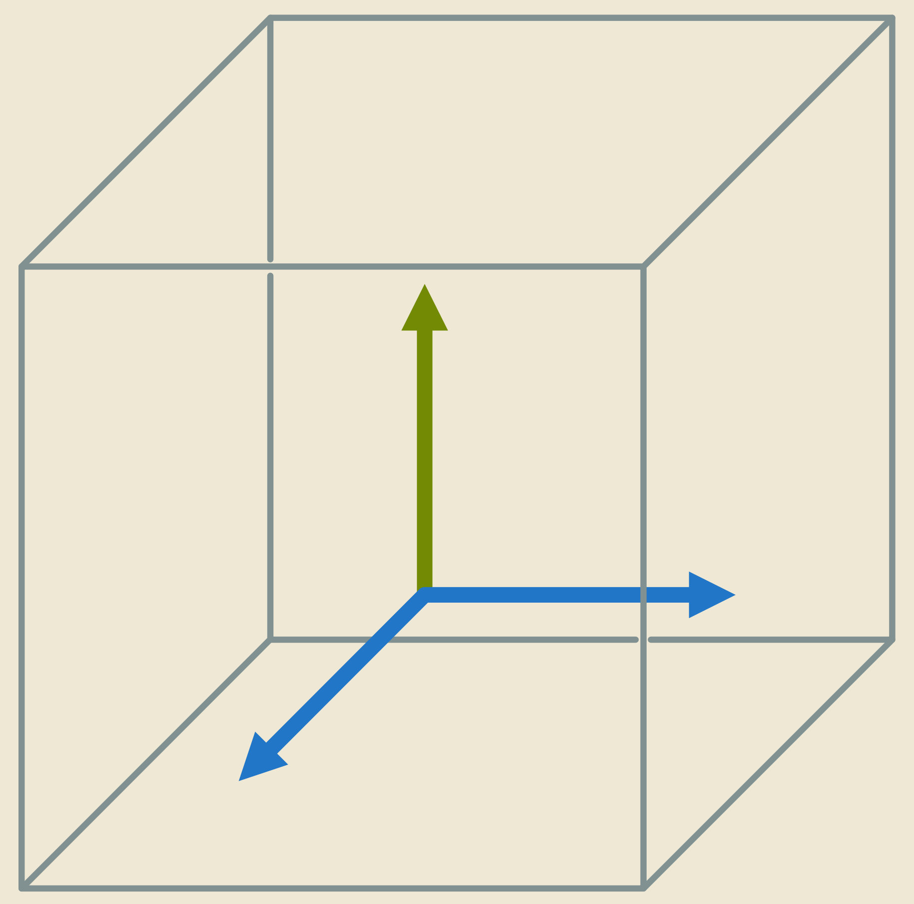
($\bar{C}^{\rho}_{\mu\nu} = \check{\Gamma}^{\rho}_{\mu\nu} + \cdots$ is neither flat nor Lorentz-metric-compatible nor torsion-free)

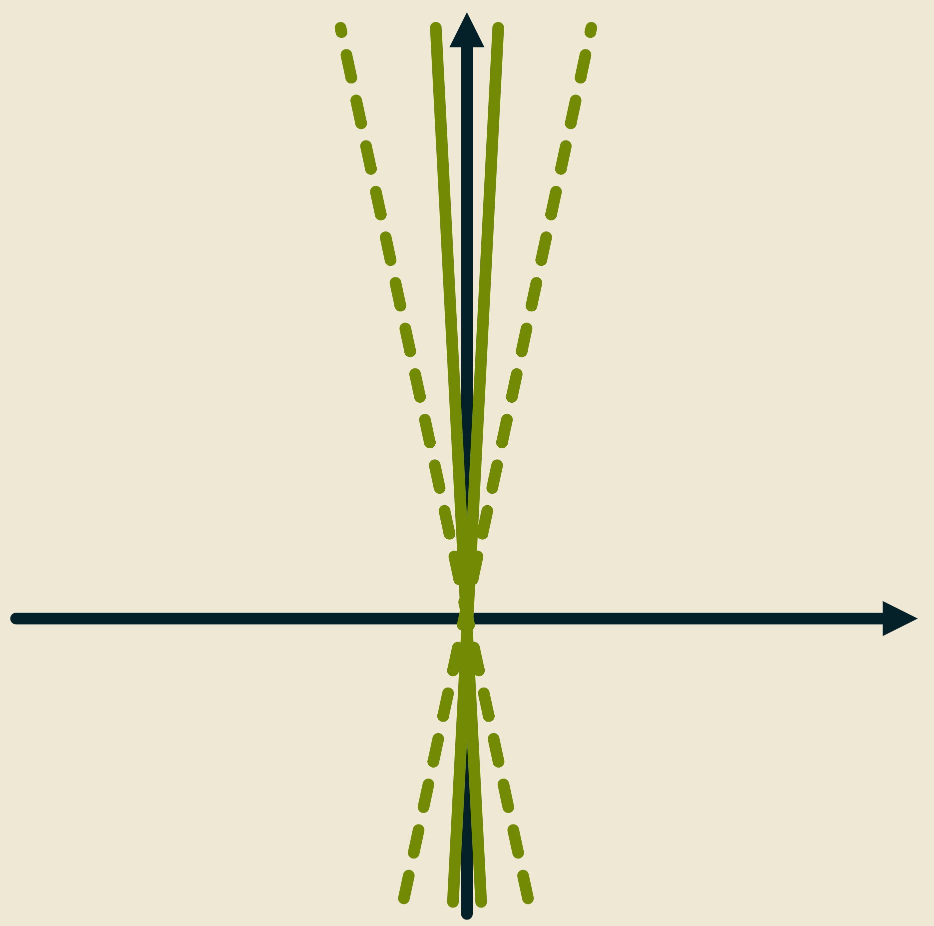
But enables us to expand the action in c^2, *Carroll geometric expansion!*

$$S = c^2 S_{\text{LO}} + S_{\text{NLO}} + \frac{1}{c^2} S_{\text{NNLO}} + \cdots$$

At leading order get timelike (or electric) Carroll gravity action

$$S_{\text{LO}} = \frac{1}{16\pi G}\int_M \left[K^{\mu\nu}K_{\mu\nu} - K^2\right] e\, d^d x$$





Carroll expansion of GR: timelike

Leading-order $c \to 0$ expansion of GR gives timelike/electric Carroll gravity action

$$S = \frac{1}{16\pi G}\int_M \left[K^{\mu\nu}K_{\mu\nu} - K^2\right] e\, d^d x,$$

Similar actions also found in [Henneaux] [Hartong] [Henneaux, Salgado-Rebodello]

EOM split into constraint and evolution equations [Hansen, Obers, GO, Søgaard] [Dautourt]

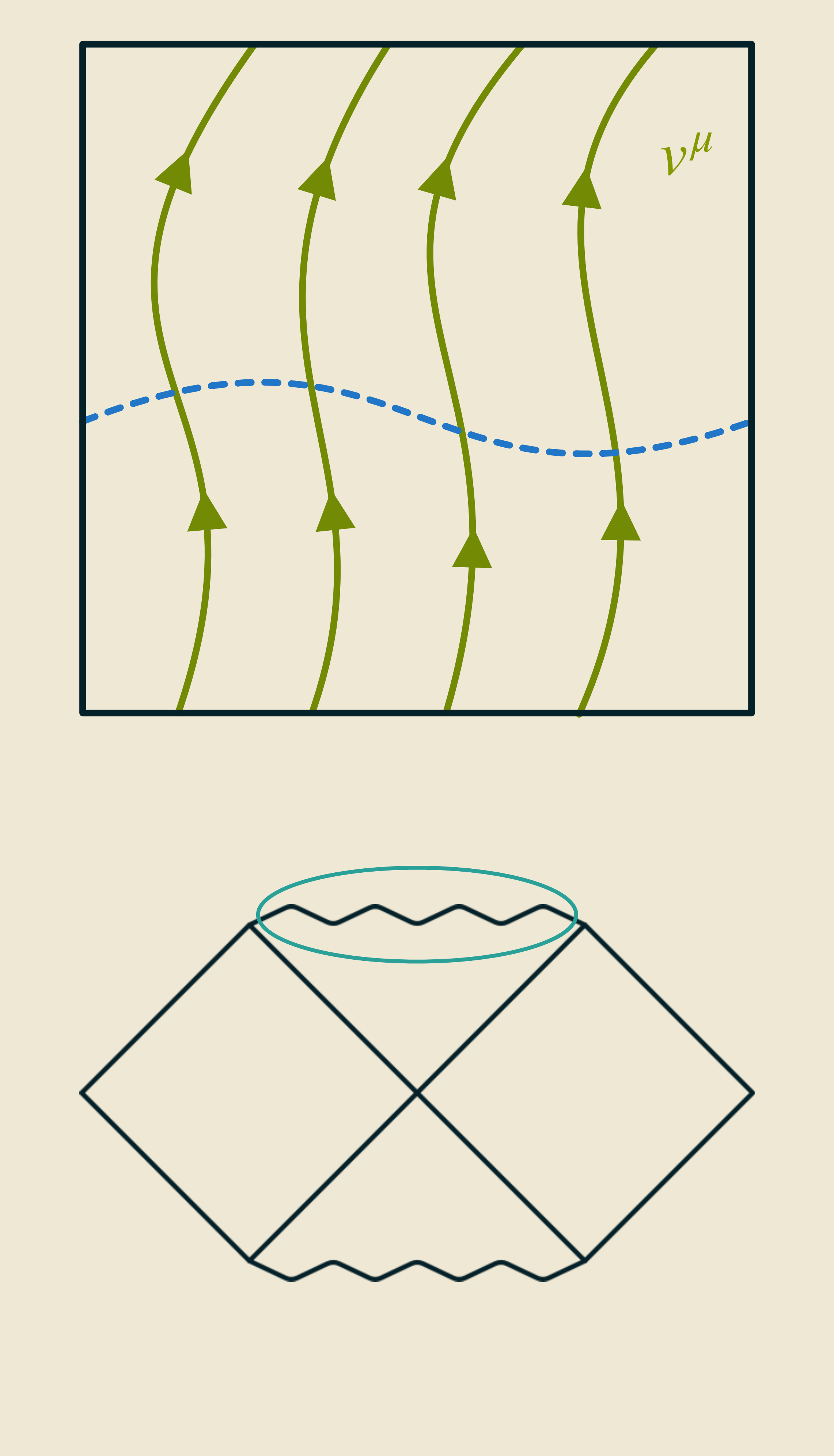
$$0 = K^{\mu\nu}K_{\mu\nu} - K^2 \qquad\qquad 0 = -R^{(3)} + K^{\mu\nu}K_{\mu\nu} - K^2$$

$$0 = h^{\rho\sigma}\tilde{\nabla}_\rho(K_{\sigma\mu} - Kh_{\sigma\mu}) \qquad\qquad 0 = h^{\rho\sigma}\nabla^{(3)}_\rho(K_{\sigma\mu} - Kh_{\sigma\mu})$$

$$\mathscr{L}_v K_{\mu\nu} = -2K_\mu{}^\rho K_{\rho\nu} + KK_{\mu\nu} \qquad\qquad \mathscr{L}_v K_{\mu\nu} = R^{(3)}_{\mu\nu} - 2K_\mu{}^\rho K_{\rho\nu} + KK_{\mu\nu} - \nabla^{(3)}_\mu a_\nu - a_\mu a_\nu$$

Evolution can be *solved analytically!* $\qquad a_\mu = v^\rho \nabla_\rho v_\mu$

Solutions include Kasner $\quad v^\mu\partial_\mu = -\partial_t\,, \quad h^{\mu\nu}\partial_\mu\partial_\nu = t^{2p_x}dx^2 + t^{2p_y}dy^2 + t^{2p_z}dz^2$



Carroll expansion of GR: spacelike

From other limit can get spacelike/magnetic Carroll gravity action

$$S = \frac{1}{16\pi G}\int_M \left[h^{\mu\nu}\tilde{R}_{\mu\nu} + \chi^{\mu\nu}K_{\mu\nu}\right] e\, d^d x,$$

Subset of full NLO action in $c \to 0$ expansion, no dynamics since $K_{\mu\nu} \sim \mathscr{L}_v h_{\mu\nu} = 0$

Projecting EOM on spatial hypersurface, constraint is now

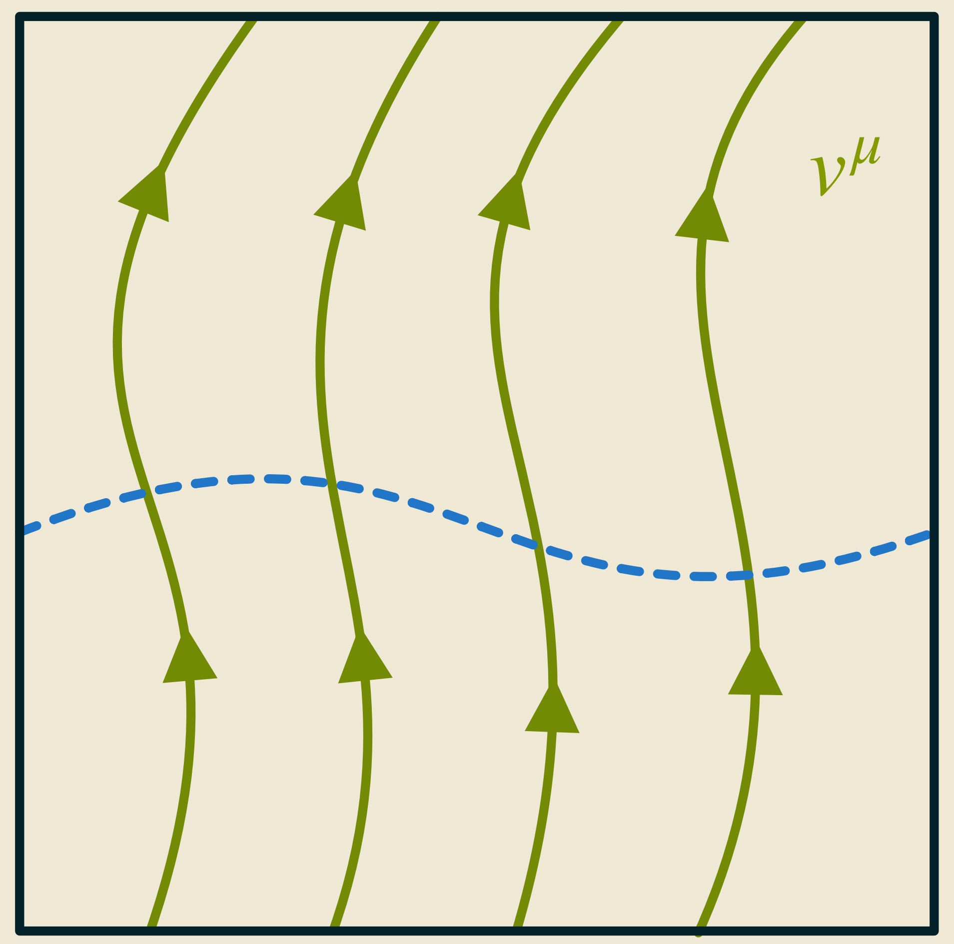
$$0 = R^{(3)}$$

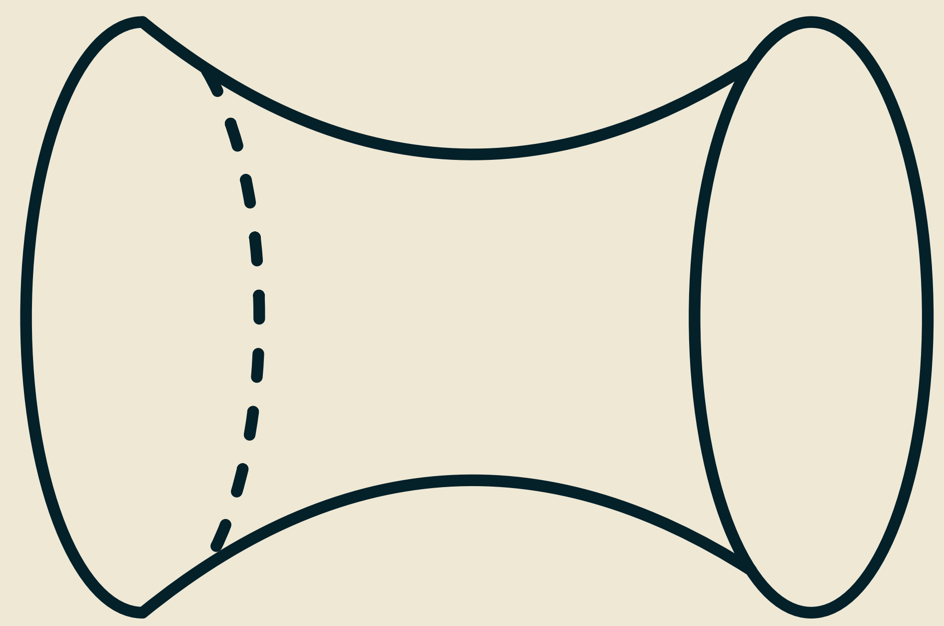
In Lorentzian EOM responsible for massive solutions $\sim -1 + \dfrac{2GM}{r}$

Indeed now find isotropic 'black hole' solution

$$v^\mu \partial_\mu = \frac{M+2r}{M-2r}\partial_t \qquad h_{\mu\nu}dx^\mu dx^\nu = \left(1 + \frac{M}{2r}\right)^4 \delta_{ij}\, dx^i dx^j$$

Dynamics in full NLO theory? Subleading corrections to BKL?





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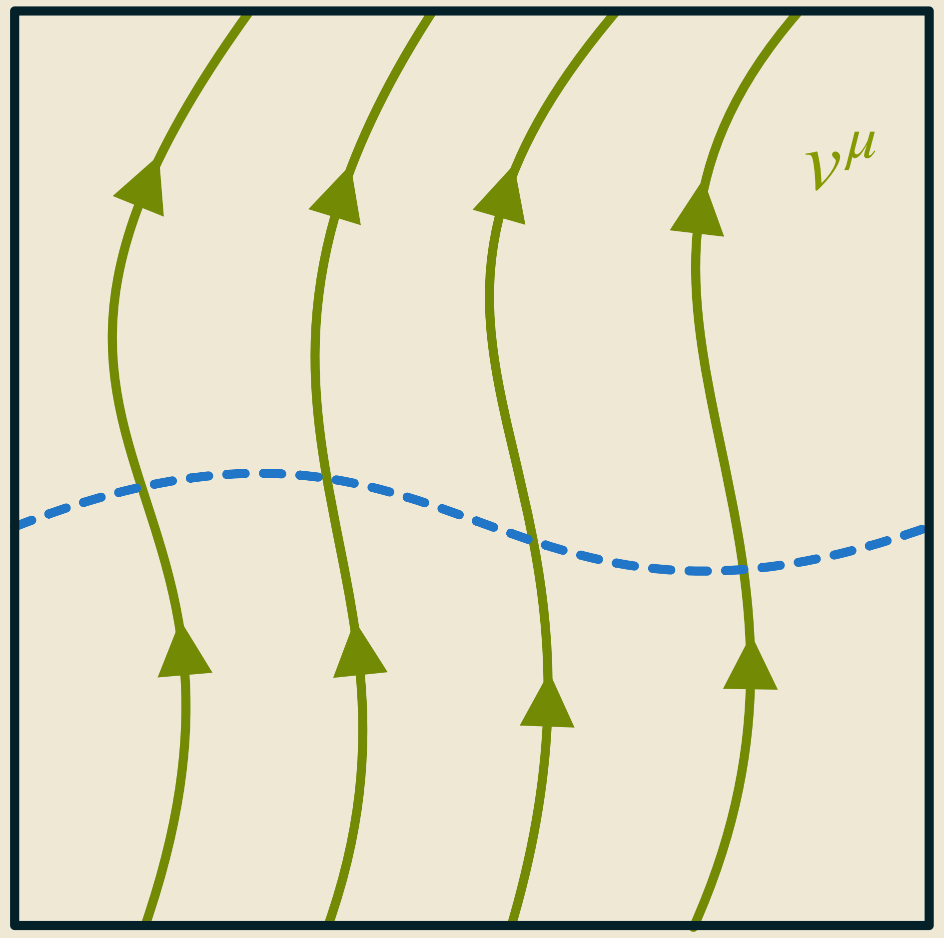

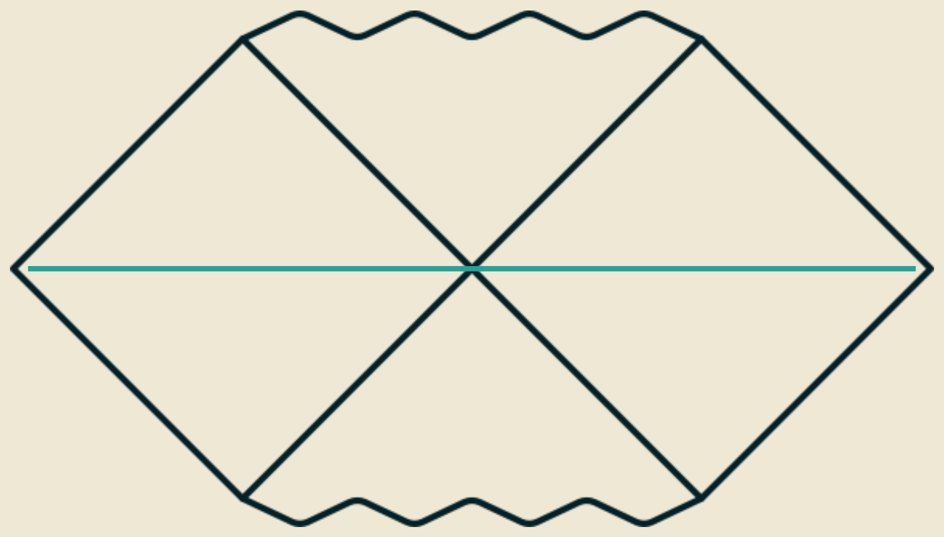

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Carroll geometry and flat holography

Holographic dual field theory for asymptotically flat spacetimes?

In 3+1 dim: BMS_4 asymptotic symmetries on $\mathscr{I}^+ \simeq \mathbb{R} \times S^2$

superrotations $z \to g(z), \quad \bar{z} \to \bar{g}(\bar{z})$

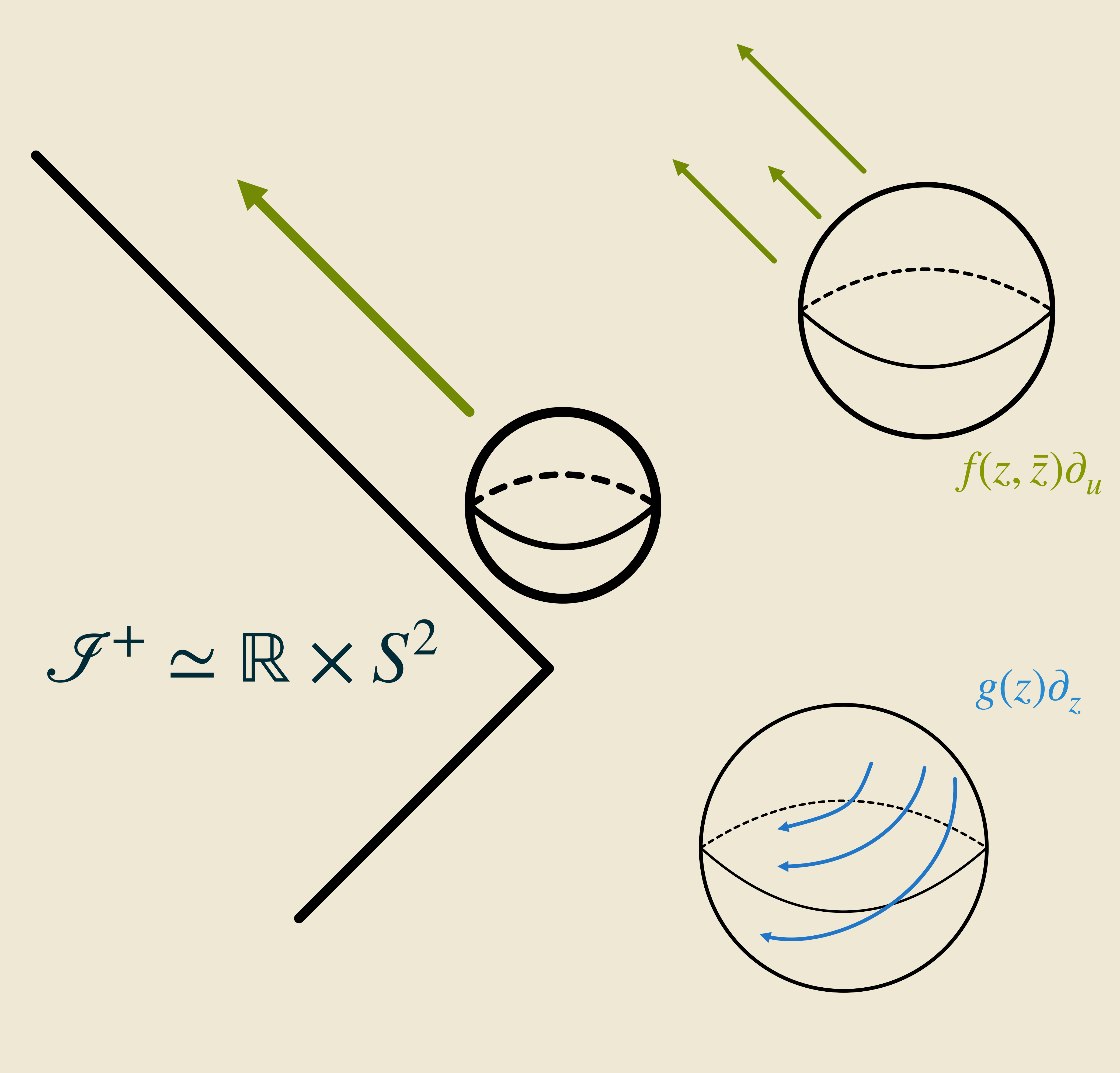
- Virasoro symmetries of CFT_2
- suggests 2d celestial CFT dual: CCFT_2

supertranslations $u \to u + f(z, \bar{z})$

- $\sim$ Carroll boosts at each $(z, \bar{z})$
- suggests 3d Carrollian CFT dual: $\mathrm{BMS}_4 \simeq \mathrm{CCar}_3$

See [Donnay, Fiorucci, Herfray, Ruzziconi] and [Bagchi, Banerjee, Basu, Dutta]

Few explicit CCFT_2 theories known,
but *can construct CCar_3 examples from $c \to 0$ limit!*



Carroll geometry and flat holography

Holographic dual field theory for asymptotically flat spacetimes?

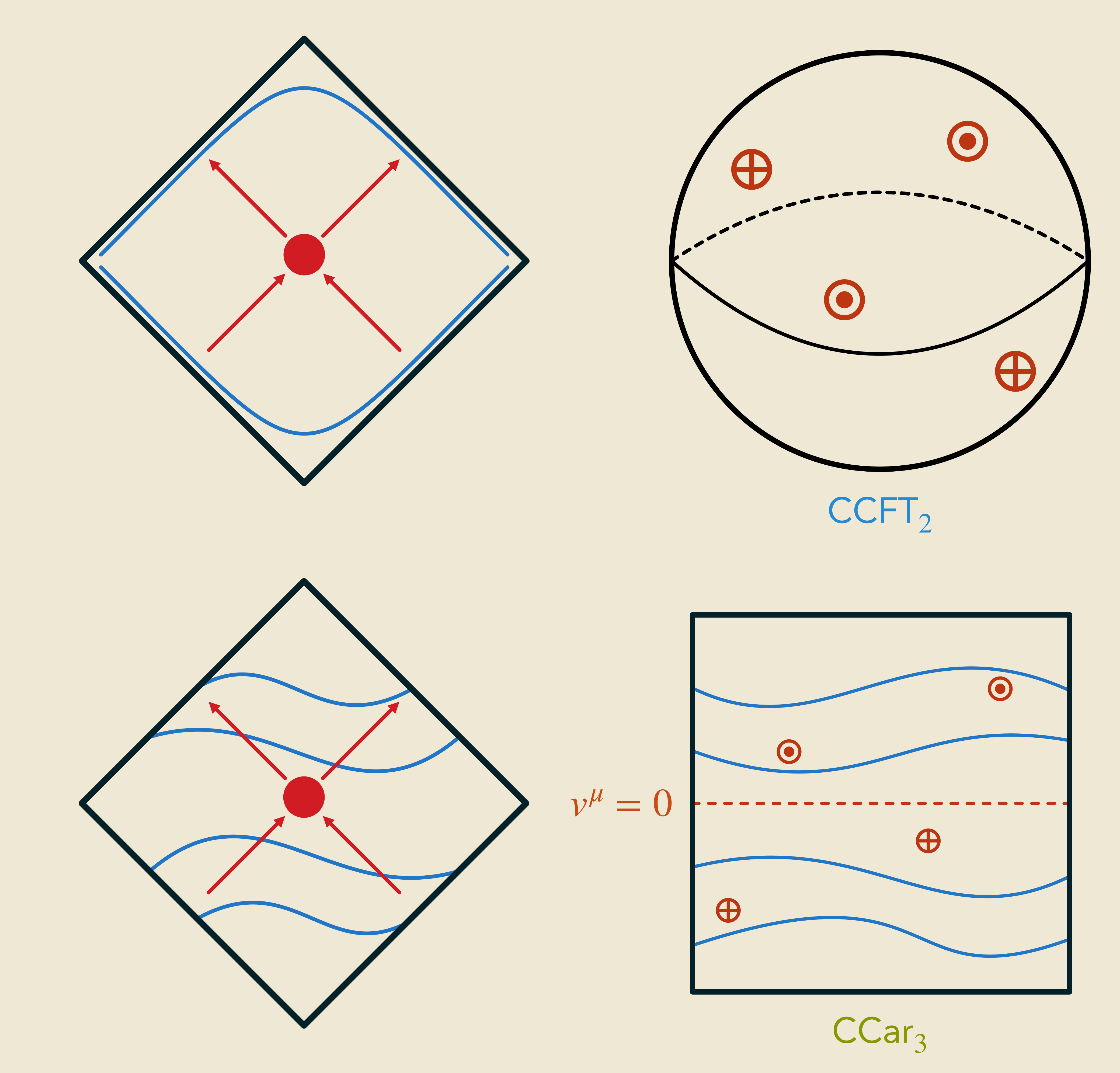
Distinct Cauchy surfaces in bulk:

Celestial CFT_2 ~ S-matrix scattering process

Conformal Carrol_3 ~ natural limit of AdS/CFT?

Related by Fourier and/or (modified) Mellin transform
[Donnay, Fiorucci, Herfray, Ruzziconi] [Bagchi, Banerjee, Basu, Dutta]

But what is overlap?

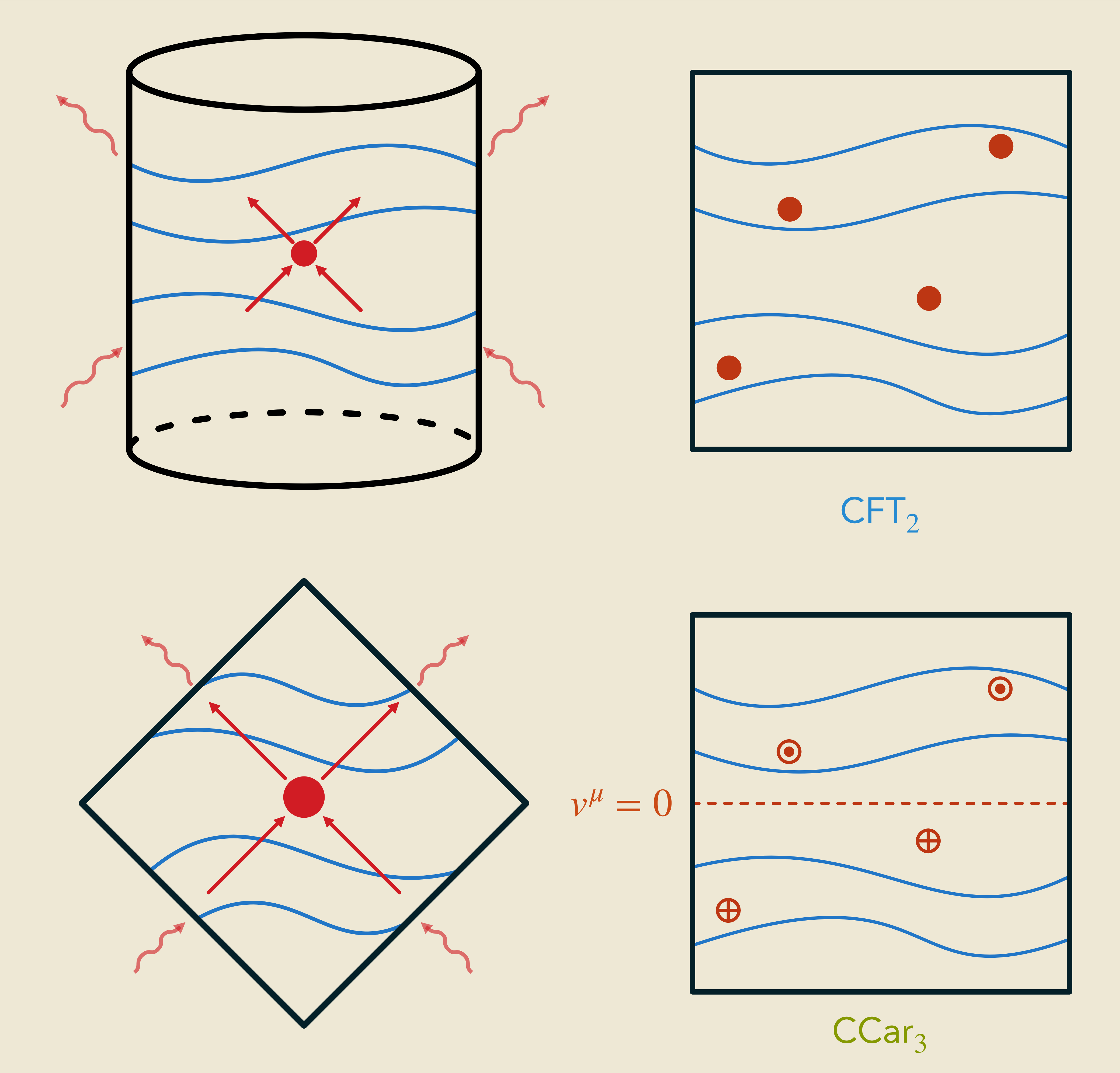


Carroll geometry and flat holography

Holographic dual field theory for asymptotically flat spacetimes?

Conformal Carrol_3 from a limit of AdS/CFT?

- Correlators arise from AdS Witten diagrams?
  [Pipolo de Gioia, Raclariu] [Bagchi, Dhivakar, Dutta]
- Need 'leaky' Λ-BMS boundary conditions in AdS
  to take limit of gravitational asymptotic phase space
  [Compère, Fiorucci, Ruzziconi]
- Hence sources in field theory!
  [Barnich, Fiorucci, Ruzziconi] [Donnay, Herfray, Fiorucci, Ruzziconi]



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Summary and outlook

Have ultra-local Carroll limits and expansion of GR

Similar field theory limits and expansion related to flat holography

Ongoing and future challenges:

- Carroll gravity at NLO and subleading BKL?
- further develop conformal Carroll $\Longleftrightarrow$ celestial CFT dictionary
- understand role of 'leakiness' in flat holography (limits)

Interpolate between full GR and non-Lorentzian limits?

Top-down flat holography from $c \to 0$ limit of AdS/CFT?