Carroll Gravity, BKL Dynamics and Holography

Gerben Oling
University of Edinburgh

Based on upcoming work with Juan Pedraza

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Outline

- Introduction: Kasner and BKL in gravity
- Carroll limits and geometry
- Mixmaster from Carroll gravity

Introduction

Kasner geometries

$$ds^2 = -\,dt^2 + t^{2p_x}\,dx^2 + t^{2p_y}\,dy^2 + t^{2p_z}\,dz^2$$

homogeneous and anisotropic solution to Einstein equations

Adding spatial curvature or matter:

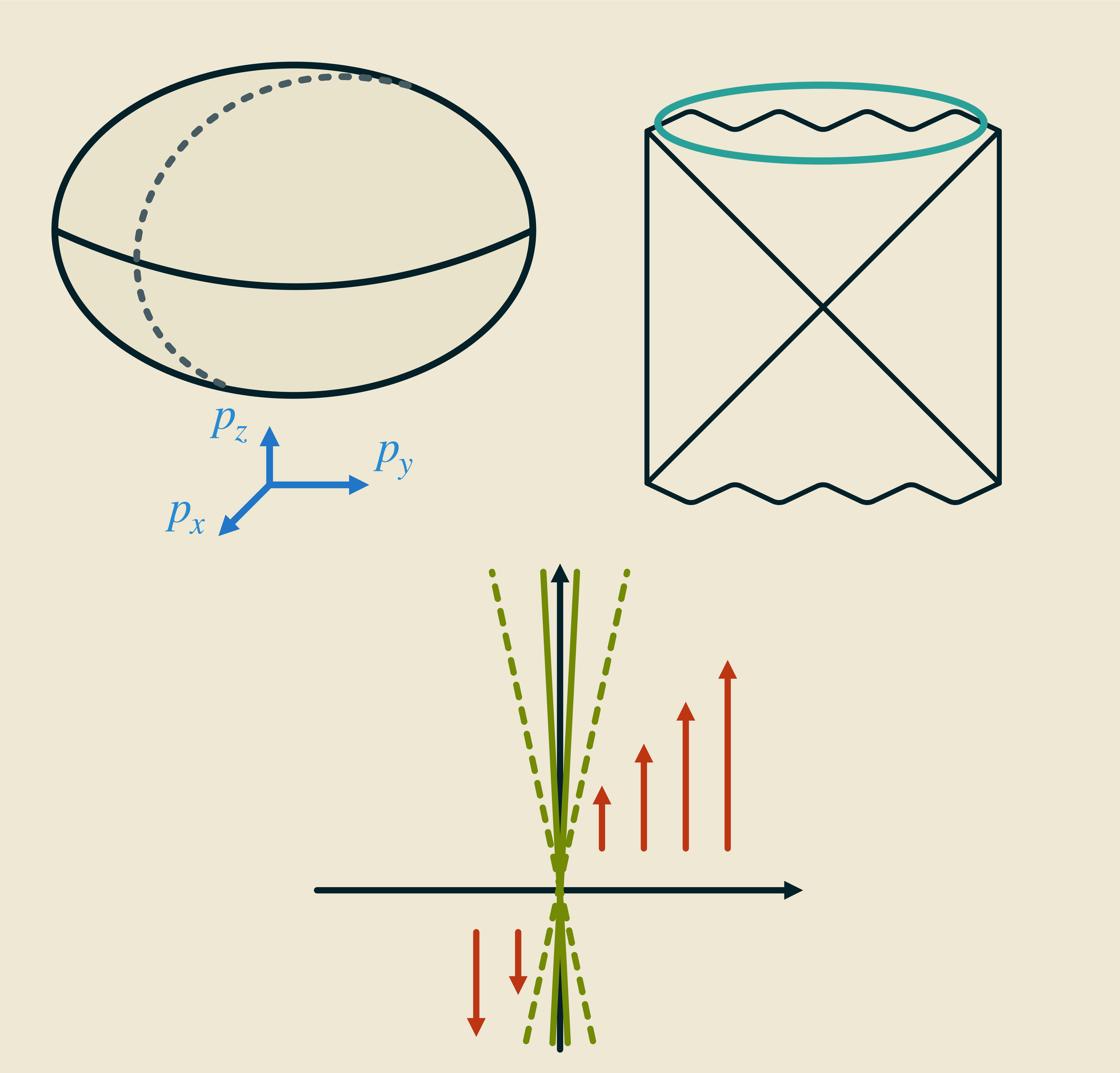
rich and possibly chaotic dynamics ('mixmaster')

BKL conjecture:

this is generic behavior of GR near spacelike singularities

Questions:

- Applications in holography?
- Near-singularity limit vs ultra-local Carroll limit?
- Mixmaster behavior from Carroll limit of GR?



Kasner geometries in GR

Take planar AdS black hole and zoom in behind horizon, $f(z) = 1 - (z/z_H)^3 \approx -(z/z_H)^3$

$$ds^2 = \frac{1}{z^2}\left[-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2 + dy^2\right]$$

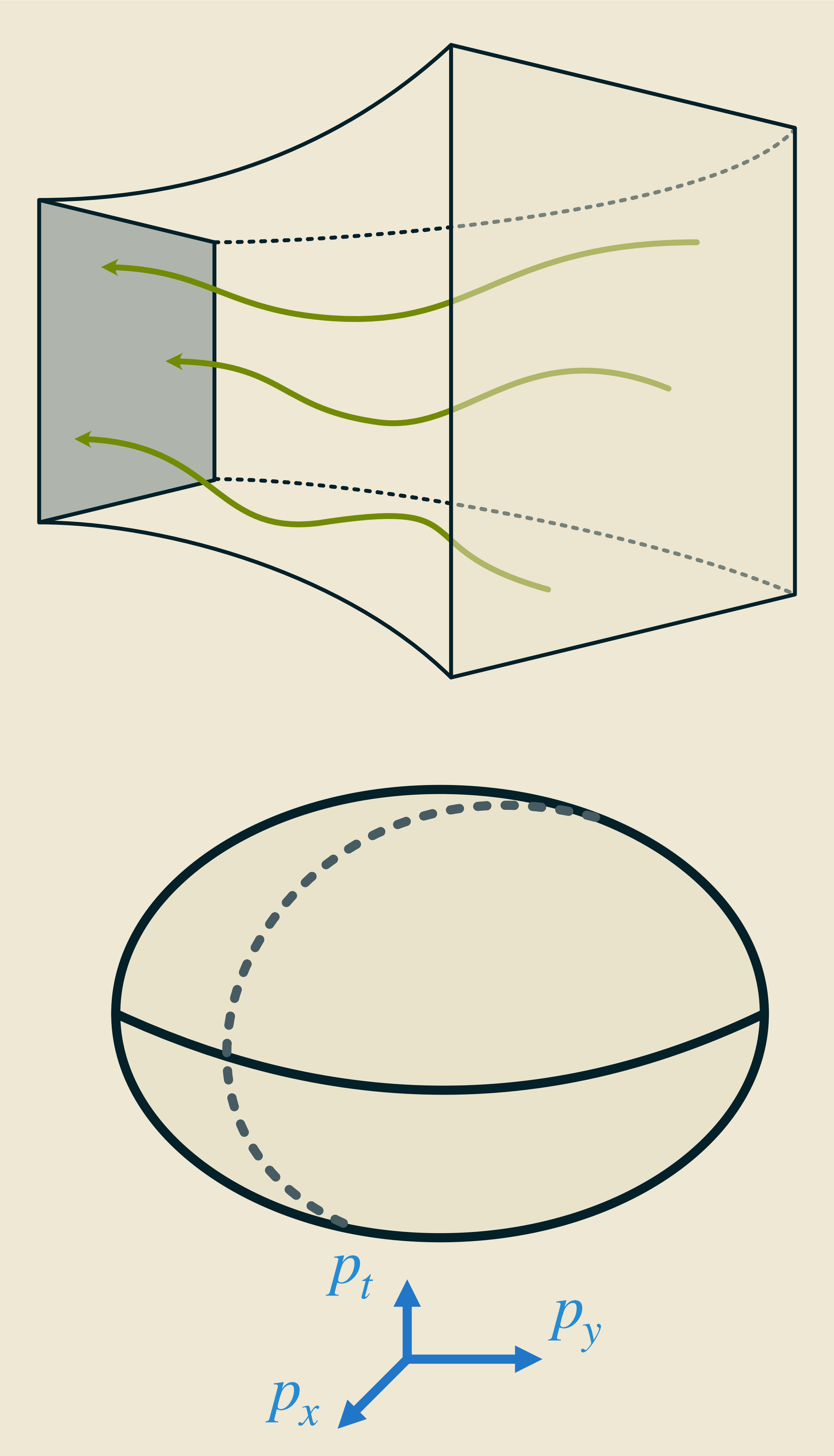
$$\approx -d\tau^2 + \#\frac{dt^2}{\tau^{2/3}} + \#\tau^{4/3}\left(dx^2 + dy^2\right)$$

where $\tau = \tau(z)$ is interior 'radial time'

Example of Kasner geometry with $p_t = -1/3$ and $p_x = p_y = 2/3$

$$ds^2 = -d\tau^2 + \tau^{2p_t}dt^2 + \tau^{2p_x}dx^2 + \tau^{2p_y}dy^2$$

Solution of vacuum Einstein equations if $\sum p_i = 1$ and $\sum (p_i)^2 = 1$



Kasner geometries in GR

Parametrize Kasner solutions using lapse $\alpha(t)$ and scaling exponents $\beta_i(t)$

$$ds^2 = -e^{-2\alpha(t)}dt^2 + e^{2\beta_x(t)}dx^2 + e^{2\beta_y(t)}dy^2 + e^{2\beta_z(t)}dz^2$$

Vacuum Einstein equations then give

$$0 = E_{tt} = \dot{\beta}^T \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \dot{\beta} = -\left(\dot{\beta}_1\right)^2 + \left(\dot{\beta}_2\right)^2 + \left(\dot{\beta}_3\right)^2$$

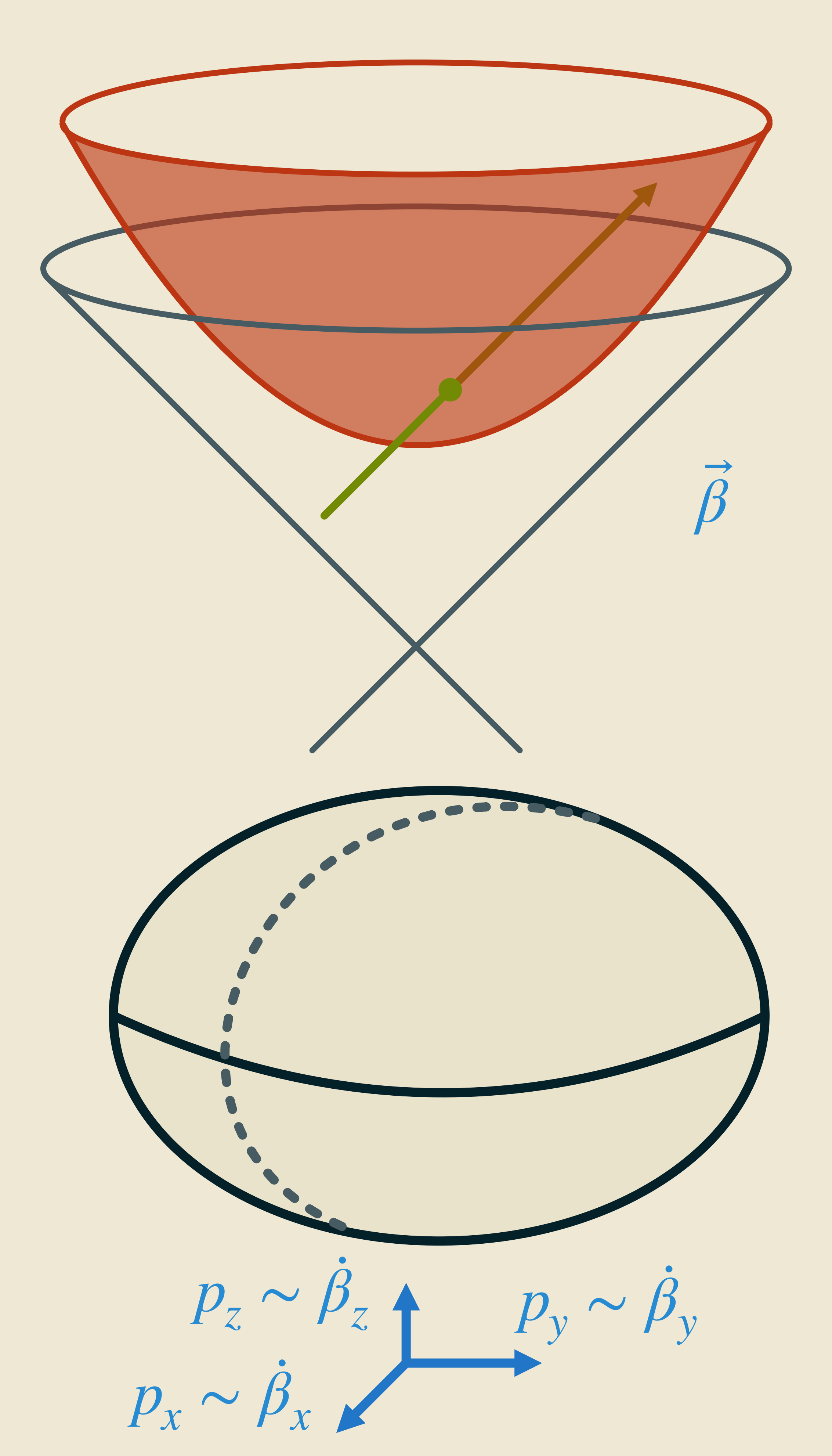
Interpret $\dot{\beta}_i(t)$ as null vector in Minkowski superspace! [Chitre] [Damour, Henneaux, Nicolai]

After reparametrization $t = t(\tau)$ spatial components $E_{ii} = 0$ give

$$0 = \ddot{\beta}_i(\tau) \qquad \Longrightarrow \qquad \beta_i = \beta_i^{(0)} + v_i \tau$$

so $\beta_i(\tau)$ parametrizes null geodesic in $\mathbb{R}^{1,2}$ superspace!

Maps to (so far simple) particle motion on future hyperboloid $\mathbb{H}^2 \subset \mathbb{R}^{1,2}$



Mixmaster dynamics in GR

Interesting dynamics from spatial curvature and/or matter coupling

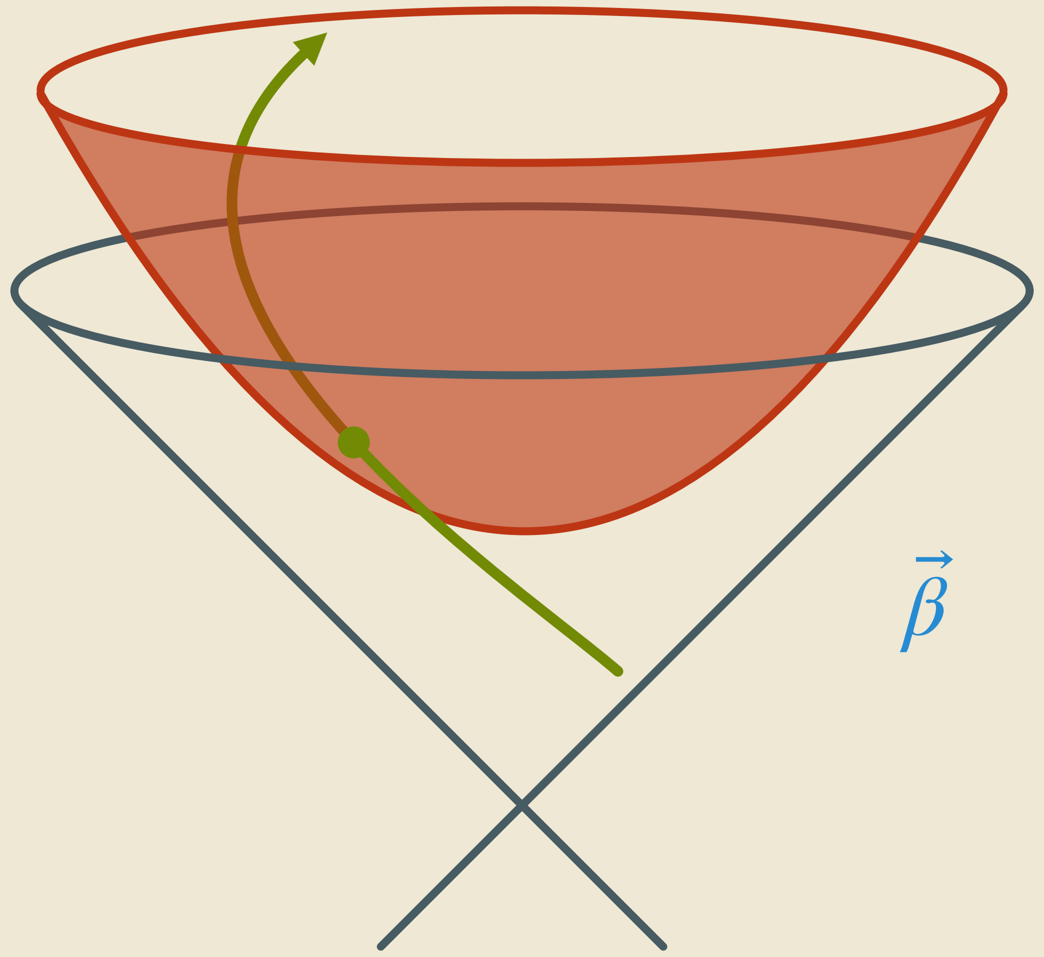
Example: $SO(3)$ group manifold ('mixmaster/Bianchi IX') [Misner]

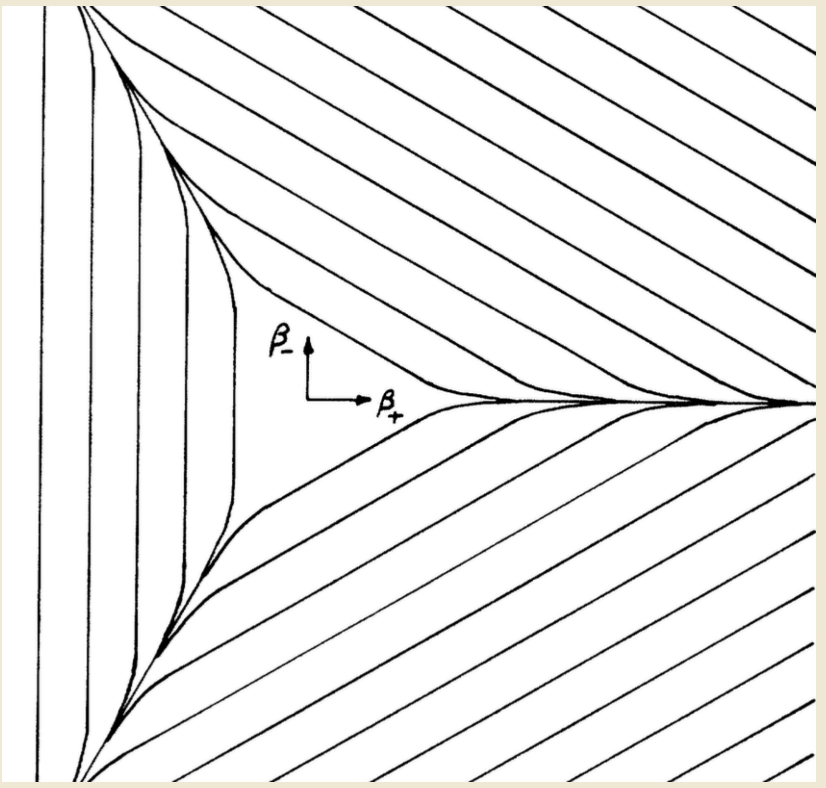
using Maurer-Cartan forms μ^i and scaling parameters $\beta_{ij}(t) = \beta_i(t)\,\delta_{ij}$ get

$$ds^2 = -\,e^{-2\alpha}dt^2 + \left(e^{2\beta}\right)_{ij}\mu^i\mu^j \qquad \Longrightarrow \qquad V(\beta) = e^{4\beta_x} + e^{4\beta_y} + e^{4\beta_z}$$

Potential modifies null geodesics, bounce around in hyperbolic triangle [Misner] [Chitre]

BKL: this kind of *chaotic behavior* is generic near singularities [Belinskii, Khalatnikov, Lifshitz]





[Misner '69]

Kasner geometries in holography

Planar AdS-RN black hole with charged massive scalar (holographic superconductor)

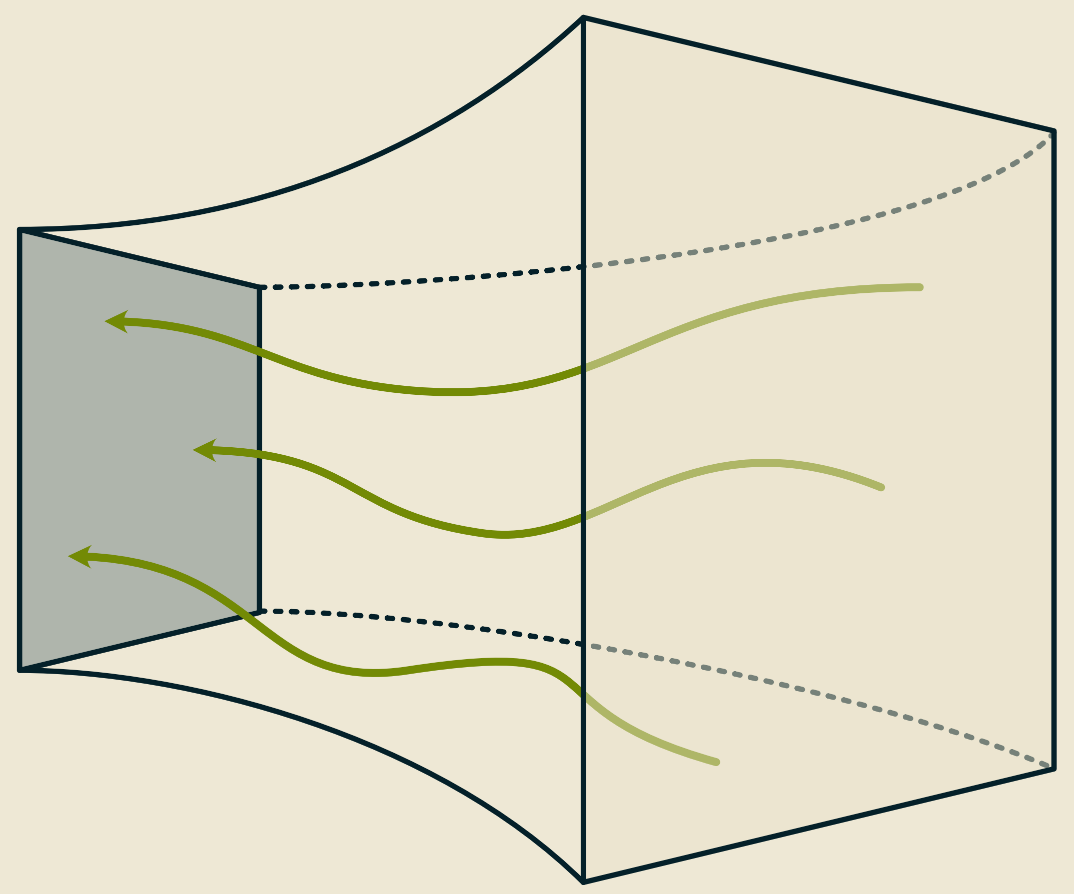
$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,(R+6) - \frac{1}{2}\int d^4x\sqrt{-g}\left(\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + D^\mu\phi\bar{D}_\nu\phi + m^2\phi^2\right)$$

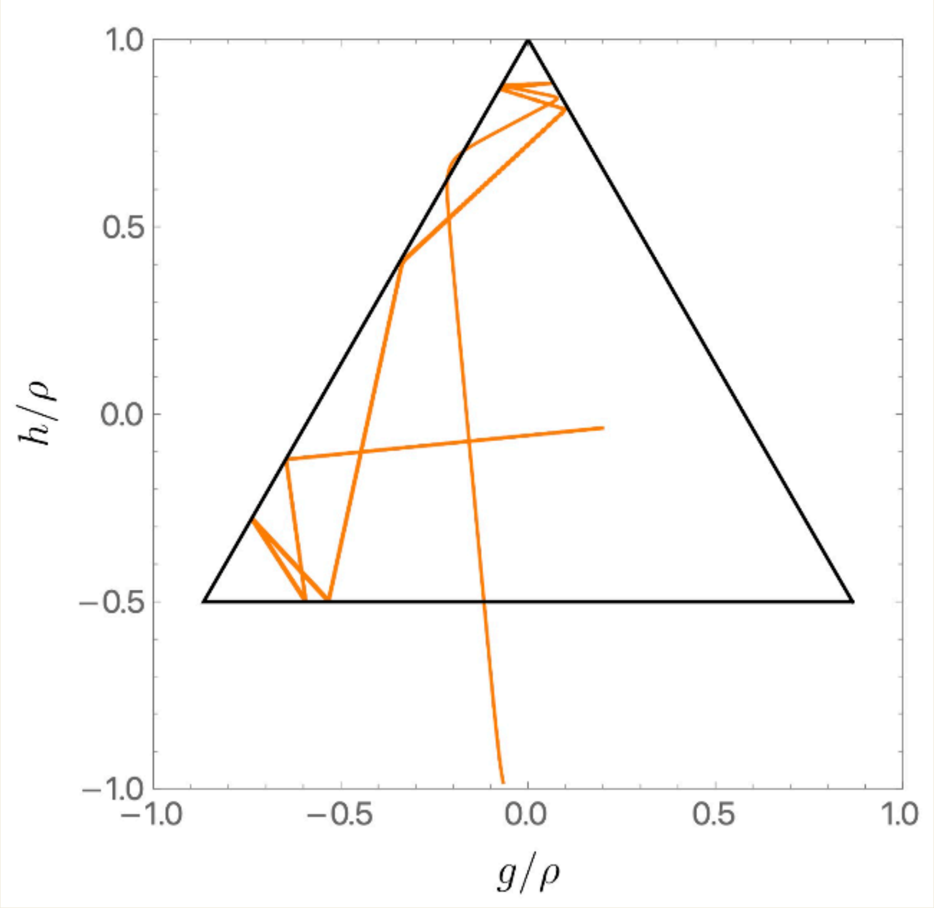
- Nontrivial dynamics behind horizon! [Hartnoll, Horowitz, Kruthoff, Santos]
- Many Kasner epochs, but eventually reaches final state

Mixmaster-style chaotic behavior obtained from three gauge fields [De Clerck, Hartnoll, Santos]

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,(R+6) - \frac{1}{2}\int d^4x\sqrt{-g}\left(\frac{1}{2}F^{\mu\nu}_{(i)}F^{(i)}_{\mu\nu} + \mu^2_{(i)}A^2_{(i)}\right)$$

- Give same hyperbolic triangle dynamics in interior!
- Many microstates after quantization?





[De Clerck, Hartnoll, Santos]

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Carroll geometry

From ‘relativistic’ Lorentz boosts

$$t \to t + \beta x, \qquad x \to x + \beta t$$

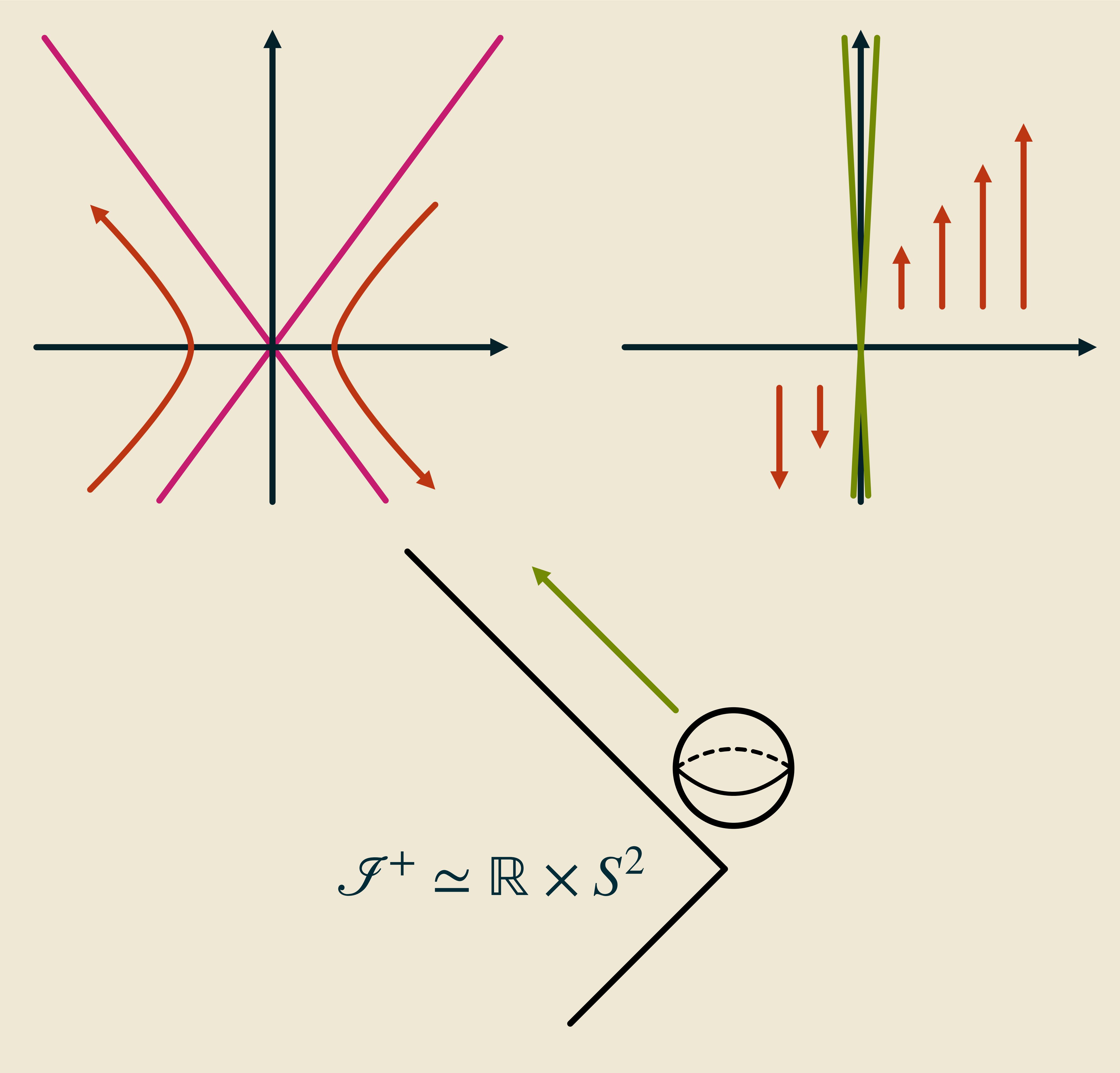
get Carroll boosts in ultra-local $c \to 0$ limit, [Levy-Leblond] [Sen Gupta]

$$t \to t + \lambda x\,, \quad x \to x \quad \text{and} \quad \partial_t \to \partial_t\,, \quad \partial_x \to \partial_x + \lambda \partial_t$$

Less obviously physical than non-relativistic $c \to \infty$ limit, but:

- appears in Lorentzian geometry on null surfaces such as $\mathscr{I}^+$
- BMS asymptotic symmetries are isomorphic to conformal Carroll algebra [Duval, Gibbons, Horvathy, Zhang]

Here: directly use ultra-local limit in GR to find BKL dynamics



Carroll geometry

Curved Carroll geometry is specified by

time vector field $v^\mu(x^\rho)$ and

spatial 'metric' $h_{\mu\nu}(x^\rho)$

Example: Kasner-type geometry

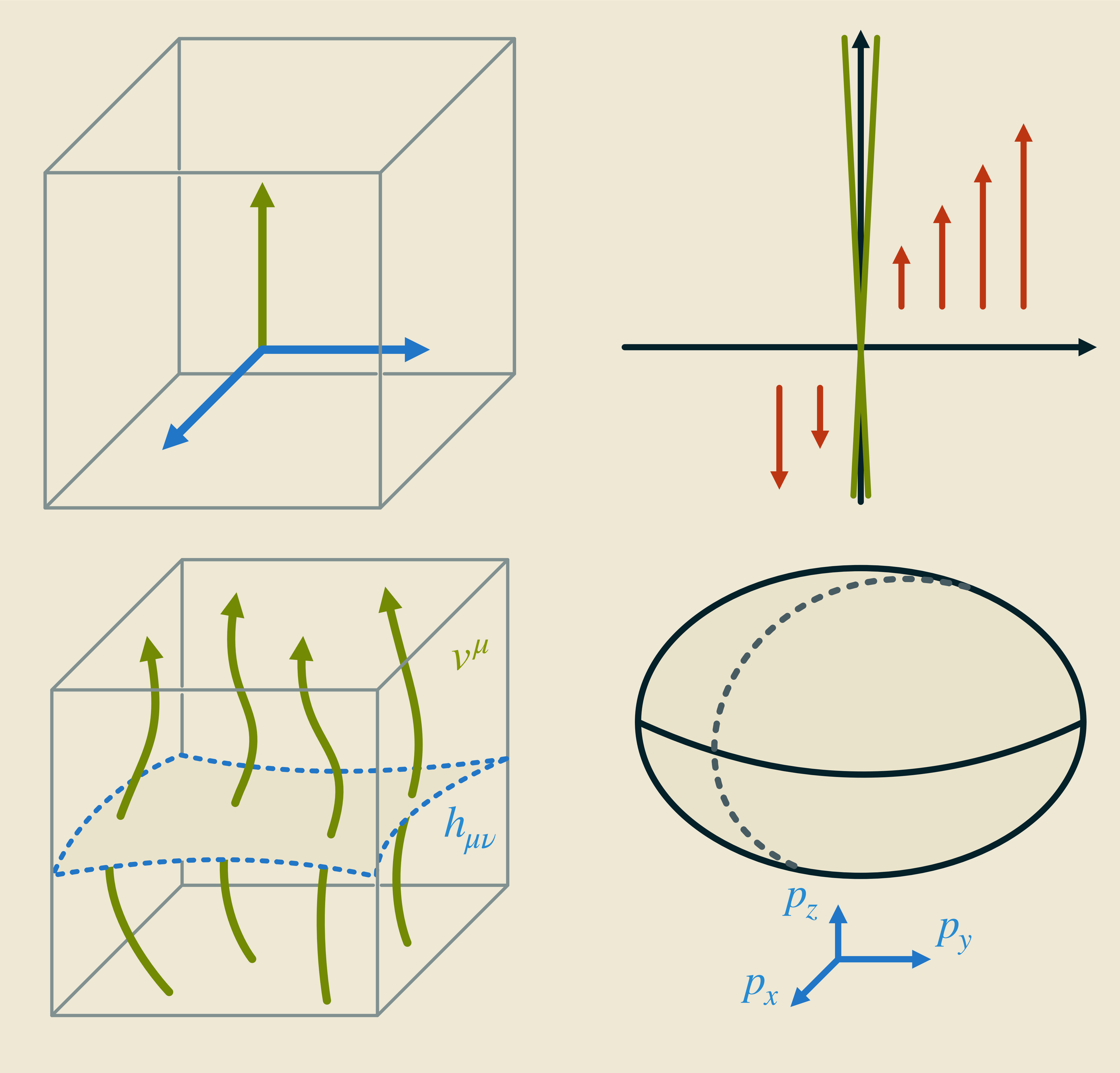
$$v^\mu \partial_\mu = -\partial_t\,, \quad h^{\mu\nu}\partial_\mu \partial_\nu = t^{2p_x}dx^2 + t^{2p_y}dy^2 + t^{2p_z}dz^2$$

Compatible connection $\tilde{\nabla}_\rho v^\mu = 0$ and $\tilde{\nabla}_\rho h_{\mu\nu} = 0$

curvature $\quad [\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\, X^\sigma = -\tilde{R}_{\mu\nu\rho}{}^\sigma X^\rho - 2\Gamma^\rho{}_{[\mu\nu]} \nabla_\rho X^\sigma$

torsion $\quad 2\tilde{\Gamma}^\rho_{[\mu\nu]} = 2h^{\rho\sigma}\tau_{[\mu}K_{\nu]\sigma}$

and extrinsic curvature $K_{\mu\nu} = -\dfrac{1}{2}\mathscr{L}_v h_{\mu\nu}$



Carroll limit of GR

Expanding GR in $c \to 0$ gives Carroll gravity at LO [Hansen, Obers, GO, Søgaard]

$$S_{\rm EH} = \frac{c^2}{2\kappa}\int_M \left[K^{\mu\nu}K_{\mu\nu} - K^2\right] e\, d^d x \quad + \;\cdots$$

(similar actions found in [Henneaux] [Hartong] [Henneaux, Salgado-Rebodello])

EOM split into constraint and evolution equations [Hansen, Obers, GO, Søgaard] [Dautcourt]

$$0 = K^{\mu\nu}K_{\mu\nu} - K^2 \qquad\qquad 0 = -R^{(3)} + K^{\mu\nu}K_{\mu\nu} - K^2$$

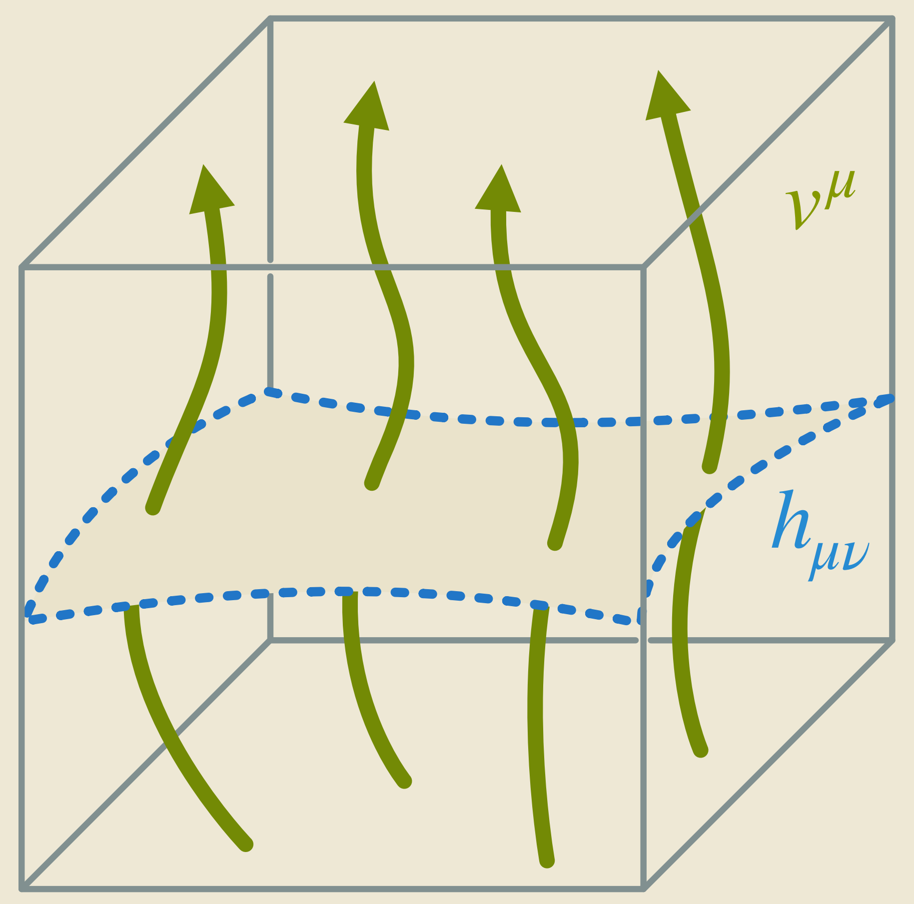
$$0 = h^{\rho\sigma}\tilde\nabla_\rho(K_{\sigma\mu} - Kh_{\sigma\mu}) \qquad\qquad 0 = h^{\rho\sigma}\nabla^{(3)}_\rho(K_{\sigma\mu} - Kh_{\sigma\mu})$$

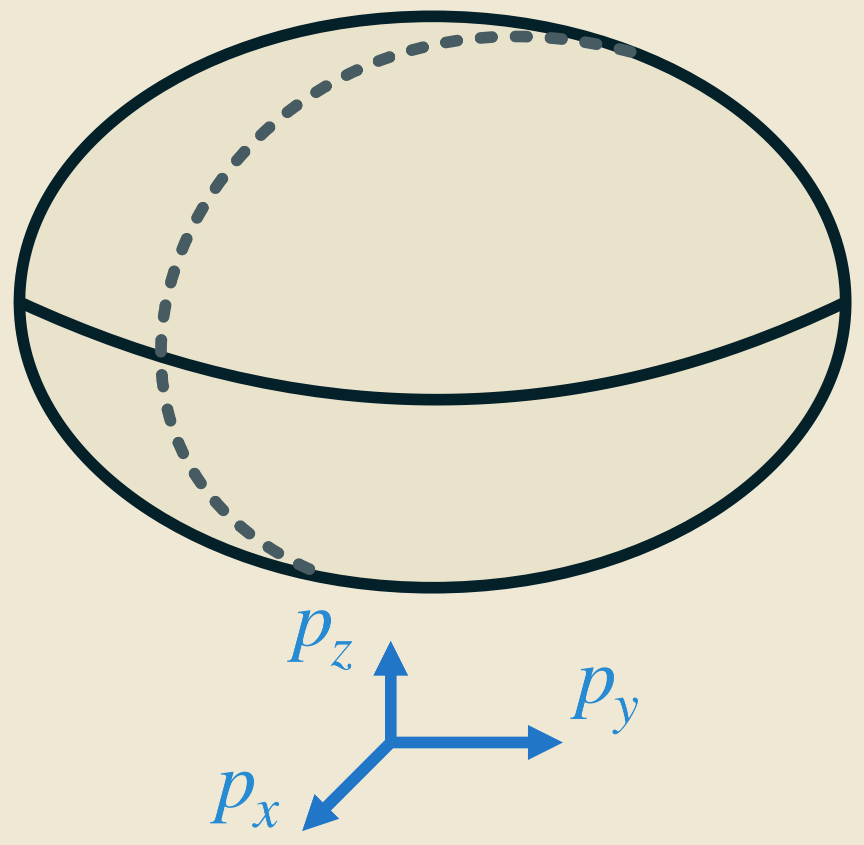
$$\mathscr{L}_v K_{\mu\nu} = -2K_\mu{}^\rho K_{\rho\nu} + KK_{\mu\nu} \qquad\qquad \mathscr{L}_v K_{\mu\nu} = R^{(3)}_{\mu\nu} - 2K_\mu{}^\rho K_{\rho\nu} + KK_{\mu\nu} - \nabla^{(3)}_\mu a_\nu - a_\mu a_\nu$$

Evolution equations are now just *ODEs!*

Solutions include Kasner geometry $\quad v^\mu\partial_\mu = -\partial_t\,, \quad h^{\mu\nu}\partial_\mu\partial_\nu = t^{2p_x}dx^2 + t^{2p_y}dy^2 + t^{2p_z}dz^2$

[Søgaard] [Henneaux] [De Boer, Hartong, Obers, Sybesma, Vandoren]





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Carroll matter coupling

Take leading-order Carroll gravity coupled to electric $U(1)^3$ Carroll YM

$$\frac{1}{2\kappa}\int d^d x\, e\left(K^{\mu\nu}K_{\mu\nu} - K^2\right) + \frac{1}{2g}\int d^d x\, e\, h^{\mu\nu} E^{(i)}_{\mu} E^{(i)}_{\nu}$$

Leads to sourced constraints and evolution equations

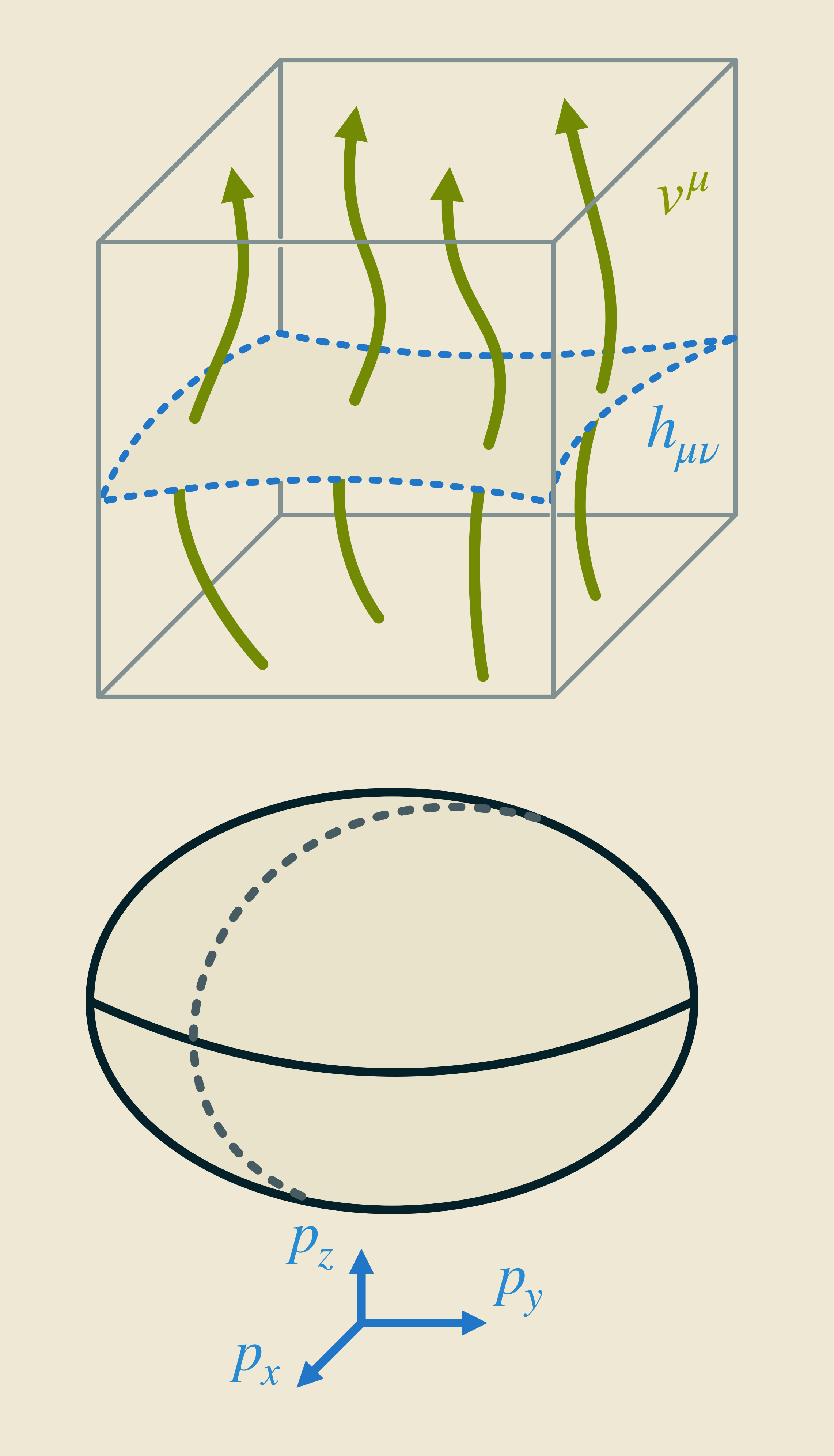
$$\frac{1}{2}\left(K^{\rho\sigma}K_{\rho\sigma} - K^2\right) = -\frac{\kappa}{2g} h^{\mu\nu} E^{(i)}_{\mu} E^{(i)}_{\mu}$$

$$-h^{\alpha\mu}h^{\rho\nu}\tilde{\nabla}_{\rho}\left(K_{\mu\nu} - K h_{\mu\nu}\right) = 0$$

$$\mathscr{L}_v K_{\mu\nu} - K K_{\mu\nu} + 2 K_{\mu}{}^{\rho} K_{\rho\nu} = \frac{\kappa}{g}\left(E^{(i)}_{\mu} E^{(i)}_{\nu} - \frac{h_{\mu\nu}}{d-1} h^{\rho\sigma} E^{(i)}_{\rho} E^{(i)}_{\sigma}\right)$$

together with gauge field EOM

$$\partial_{\mu}\left(e\, v^{[\mu} h^{\nu]\rho} E^{(i)}_{\rho}\right) = 0$$



Mixmaster from LO Carroll gravity

With homogeneous ansatz, EOM of Carroll gravity coupled to $U(1)^3$ gauge fields give

$$\ddot{\beta}_i = \frac{\kappa}{2g}\left(1 - \partial_{\beta_i}\right) V(\beta)$$

where potential gives mixmaster triangle motion for exponents [GO, Pedraza]

$$V(\beta) = (\phi_x)^2\, e^{\sqrt{2/3}\left(\bar{\beta}_0 - \bar{\beta}_1 - \sqrt{3}\bar{\beta}_2\right)} + (\phi_y)^2\, e^{\sqrt{2/3}\left(\bar{\beta}_0 - \bar{\beta}_1 + \sqrt{3}\bar{\beta}_2\right)} + (\phi_z)^2\, e^{\sqrt{2/3}\left(\bar{\beta}_0 + 2\bar{\beta}_1\right)}$$

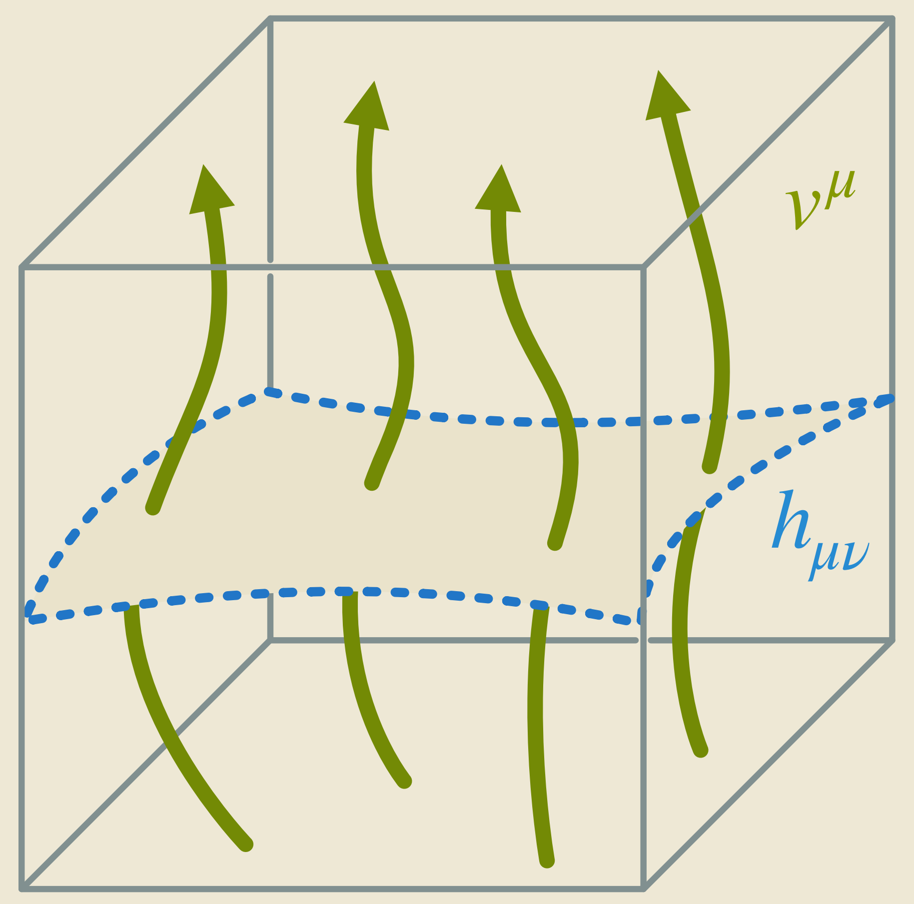
Reproduce holographic setup of [De Clerck, Hartnoll, Santos '23] using LO Carroll gravity

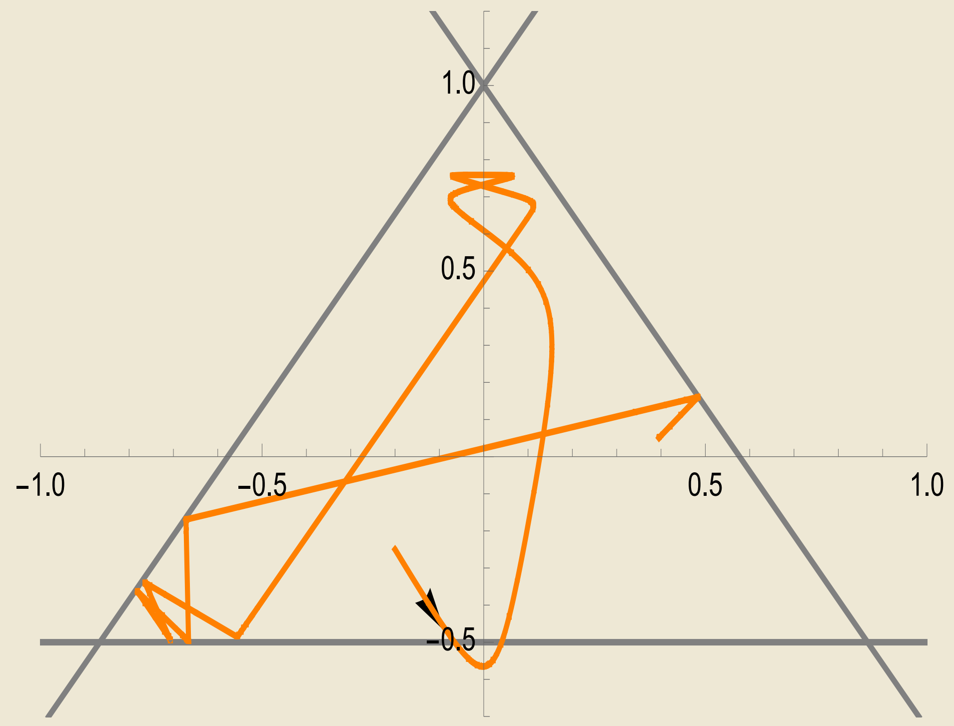
Generalize it: in LO Carroll gravity evolution equation is always ODE

still get solvable models even *without spatial homogeneity!*

Diving inside holographic superconductors without translation symmetry?

Relate to chaos at horizon using Carroll expansion?





[GO, Pedraza]

Summary and outlook

Close relation between BKL/mixmaster and Carroll expansion of GR

More models easily accessible from Carroll limits/expansion

Off-shell separation of ultralocality limit and strong gravity limit

Work in progress:

- BKL-type behavior in AdS/CFT with spatial inhomogeneity?
- 'Standard' mixmaster from $R^{(3)}$ in NLO Carroll gravity

Future:

- Expand all the way to horizon? Relation to 'redshift' chaos?
- Contact with mathematical GR literature on BKL?
- Lessons about singularities in AdS/CFT?