Expanding General Relativity in the Speed of Light

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NORDITA
The Nordic Institute for Theoretical Physics

Outline

- Introduction
- Newton-Cartan geometry
- Non-relativistic expansion of GR
- Carroll geometry
- Ultra-local expansion of GR

Why not relativistic?

What's wrong with Lorentzian symmetries?
Nothing, but *string theory is hard!*

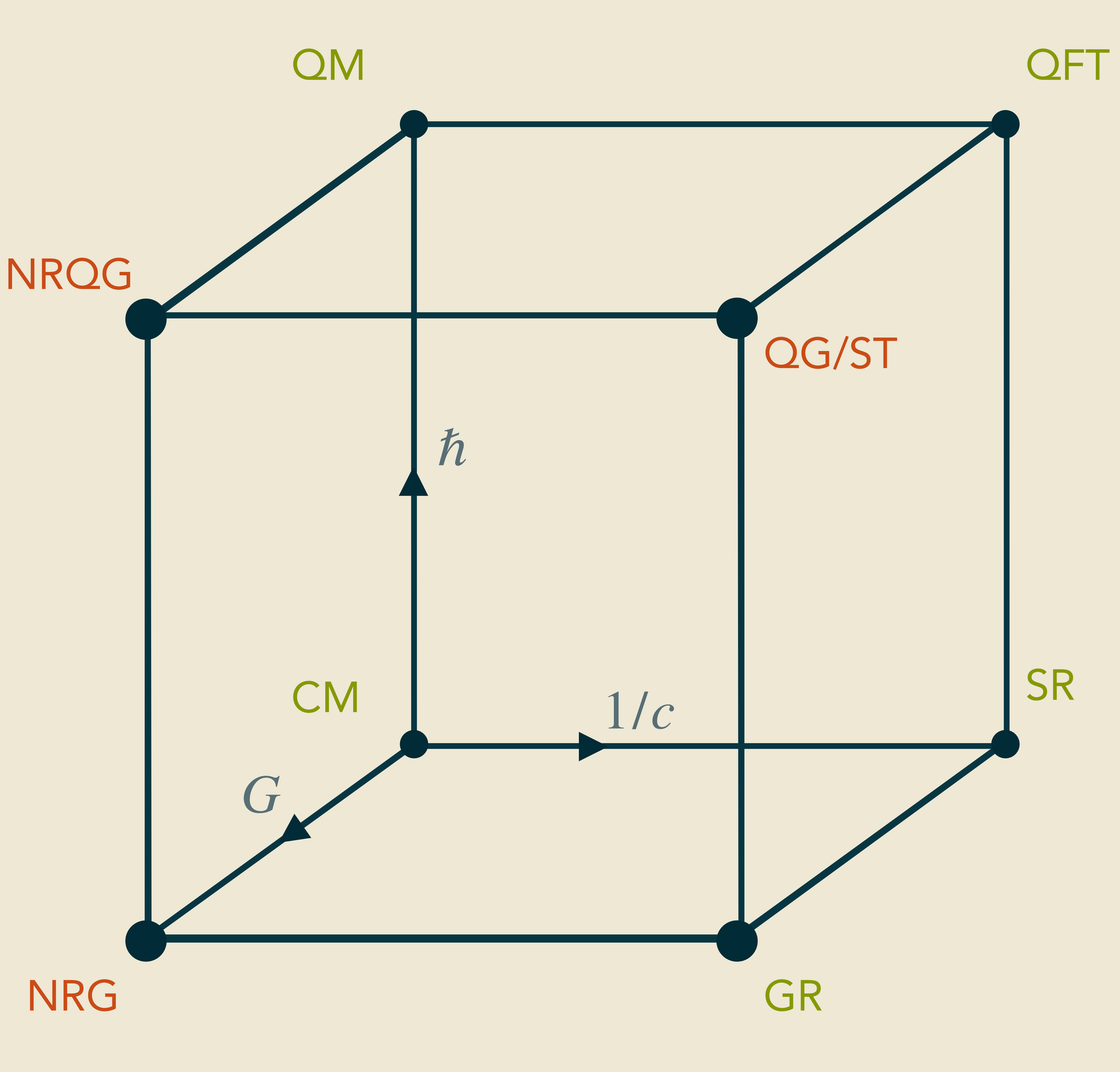
My original motivation: holography

- dual models for non-relativistic strongly-coupled matter
- break Lorentzian symmetries using background fields [Taylor]
- intrinsic non-relativistic approach?

Related: non-relativistic strings and quantum gravity [Gomis, Ooguri] [Danielsson, Guijosa, Kruczenski]

- decoupling limit of string theory
- non-relativistic spectrum
- easier worldsheet theory?

See also our recent review on non-relativistic strings [GO, Yan]



Why not relativistic?

What's wrong with Lorentzian symmetries?
Nothing, but *general relativity is also hard!*

We know how Einstein gravity contains Newtonian gravity,

$$g_{00} = -(1 + 2\Phi), \quad v/c \ll 1, \quad \text{weak coupling}$$

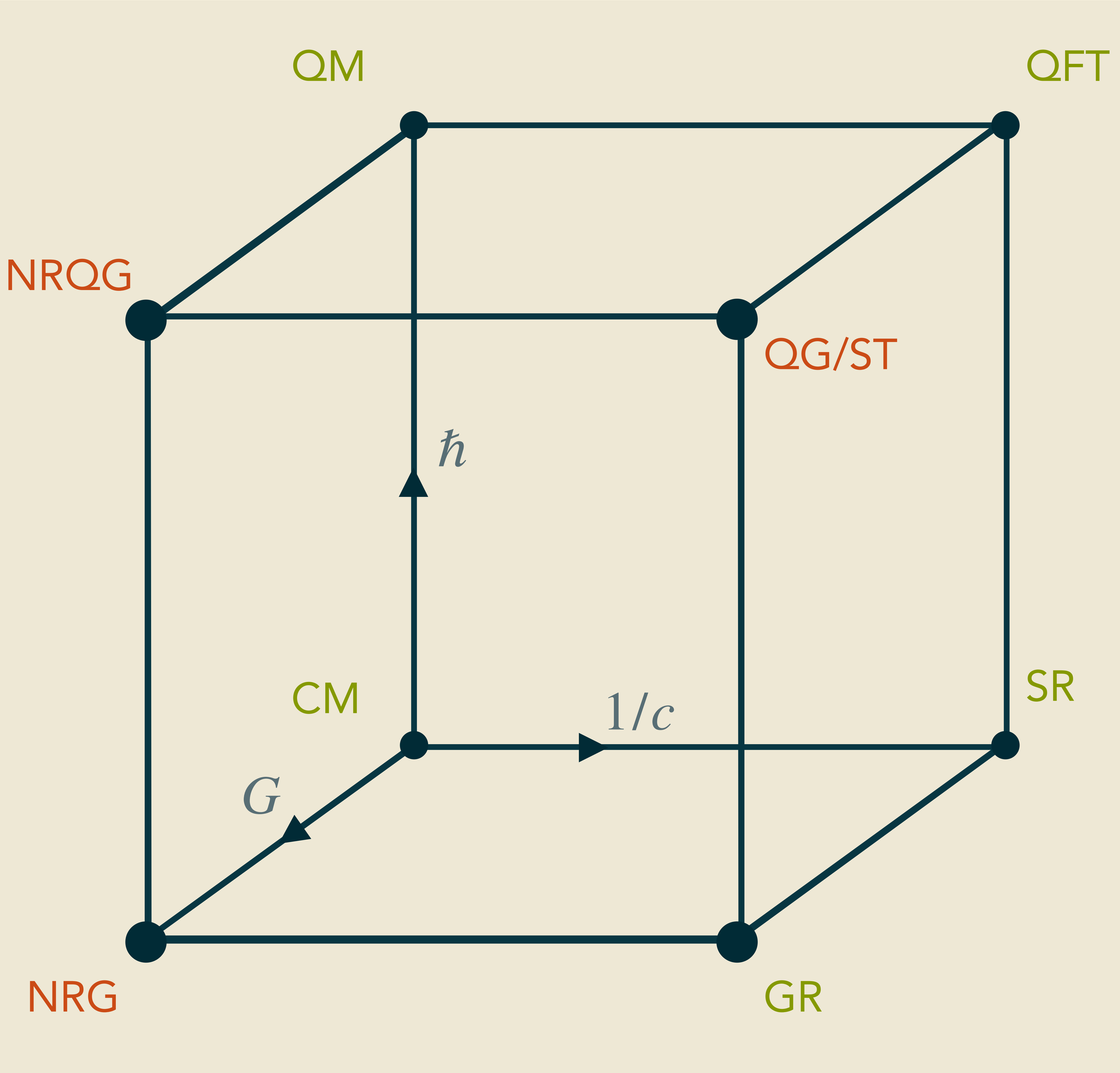
but where is the geometry? Not covariant! Galilean symmetries?

$\Longrightarrow$ Newton-Cartan geometry! [Cartan] [Künzle] [Dautcourt] …

Now understand better [Van den Bleeken] [Hansen, Hartong, Obers]

- how Newton-Cartan geometry arises from Lorentzian
- how Newtonian gravity arises from GR
- weak coupling and low velocity are independent

Main tool: covariant expansion of geometry in powers of c
around $c \to \infty$ (Galilean) and $c \to 0$ (Carroll)



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Newton-Cartan geometry

Are used to `relativistic' Lorentz boosts

$$t \to t + \beta x, \qquad x \to x + \beta t$$

Non-relativistic limit $c \to \infty$ gives Galilean boosts

$$t \to t\,, \quad x \to x + \lambda t\,, \quad \text{and} \quad \partial_t \to \partial_t + \lambda \partial_x\,, \quad \partial_x \to \partial_x$$

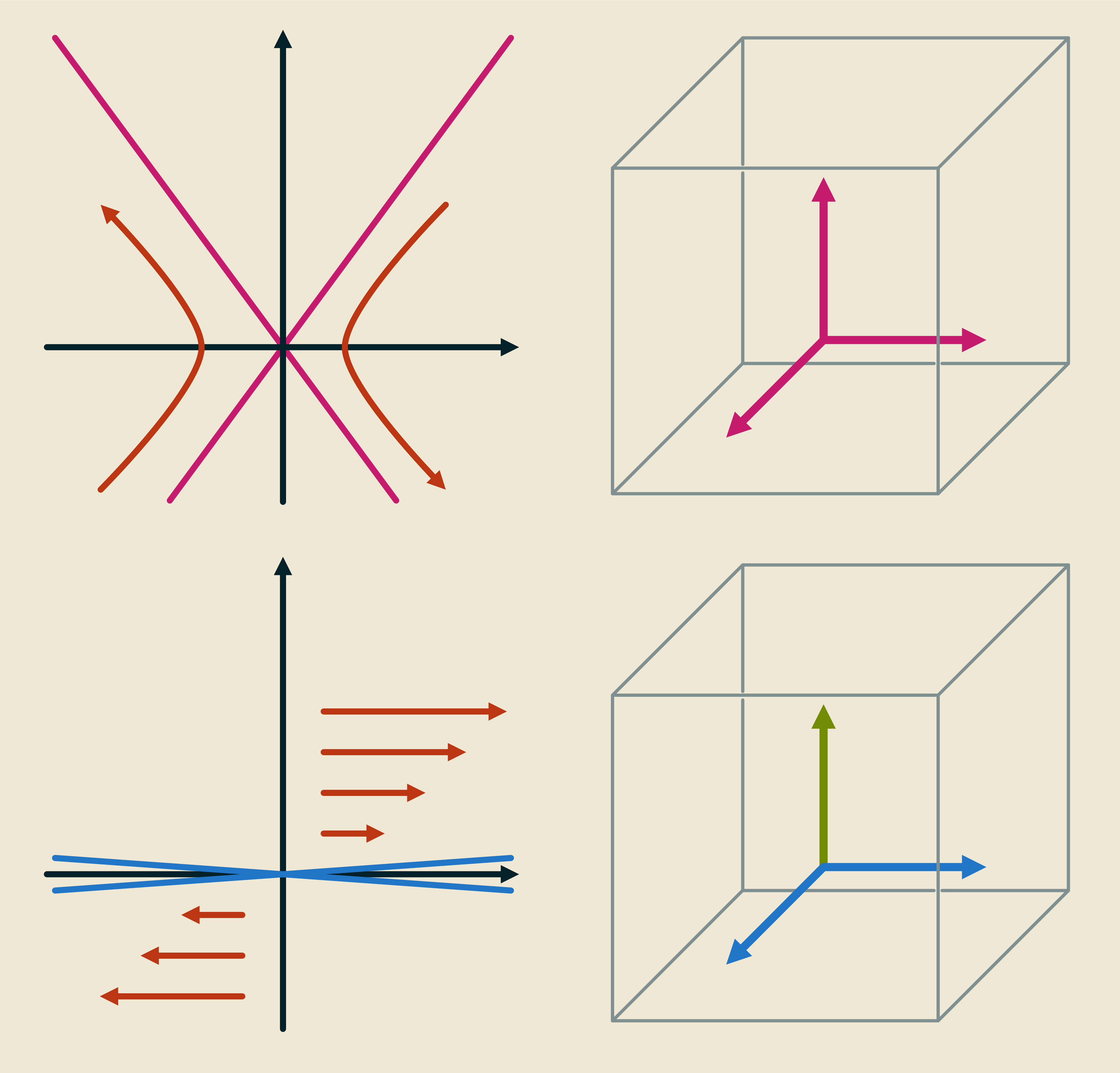
Curved extension: not Lorentzian $g_{\mu\nu}(x^\rho)$ but Newton-Cartan, clock one-form $\tau_\mu(x^\rho)$ and spatial metric $h^{\mu\nu}(x^\rho)$

Complement with inverse $v^\mu(x^\rho)$ and $h_{\mu\nu}(x^\rho)$, satisfy

$$v^\mu h_{\mu\nu} = 0, \quad \tau_\mu h^{\mu\nu} = 0, \quad v^\mu \tau_\mu = -1, \quad \delta^\mu_\nu = -v^\mu \tau_\nu + h^{\mu\rho} h_{\rho\nu}$$

Transform under local Galilean boosts $\lambda_\mu(x^\rho)$ as

$$\delta_\lambda v^\mu = \lambda^\mu\,, \qquad \delta_\lambda h_{\mu\nu} = \lambda_\mu \tau_\nu + \tau_\mu \lambda_\nu$$



Newton-Cartan geometry

Newton-Cartan: clock one-form $\tau_\mu(x^\rho)$ and spatial metric $h^{\mu\nu}(x^\rho)$

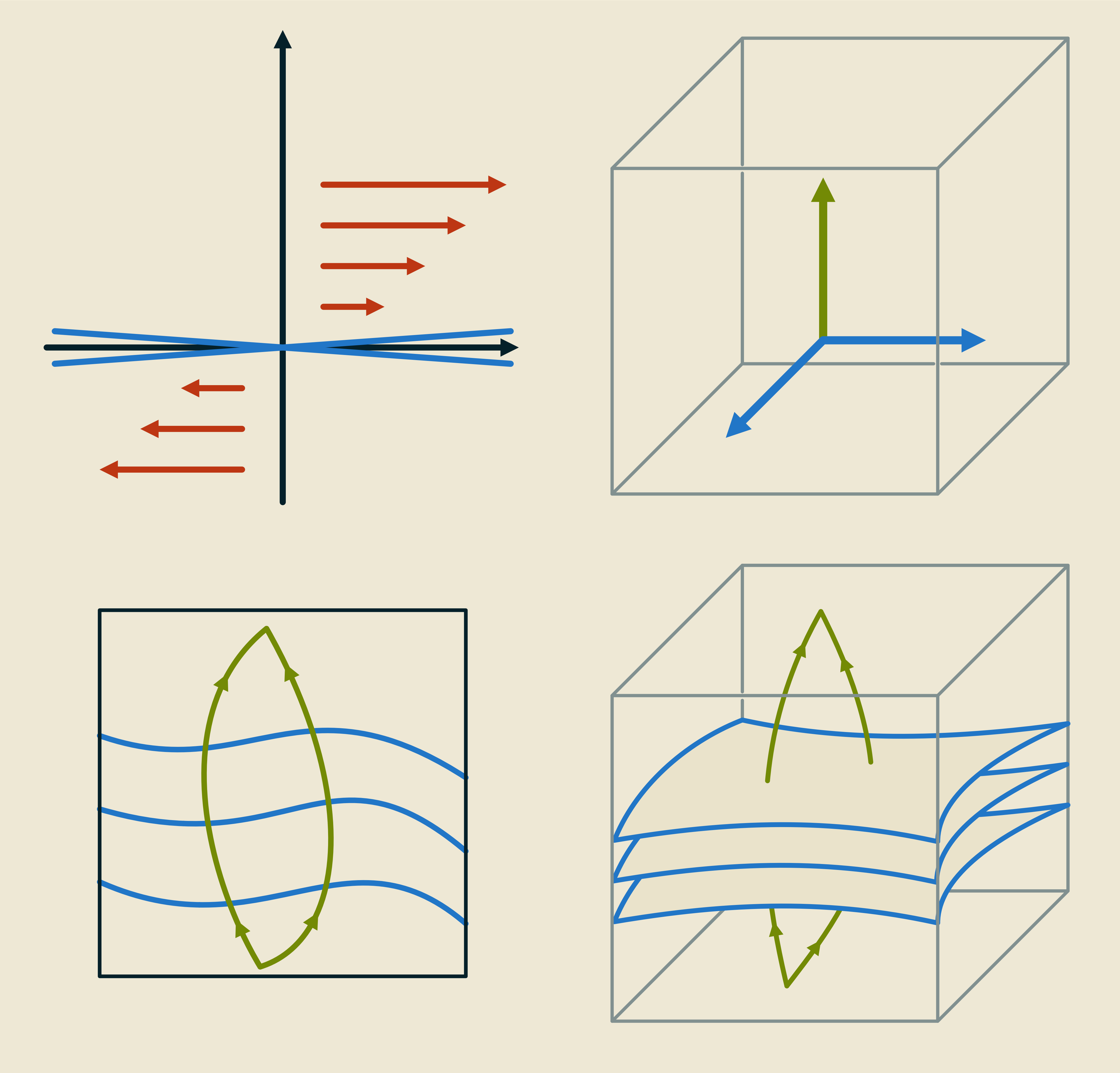
Clock form gives space-time structure:

- if $d\tau \neq 0$ but $\tau \wedge d\tau = 0$ get spatial foliation
- if $d\tau = 0$ have $\tau = dt$, so *absolute time* (path-independent)

Natural connection $\check{\Gamma}^\rho_{\mu\nu} = -v^\rho \partial_\mu \tau_\nu + \frac{h^\rho \sigma}{2}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right)$

- is metric-compatible: $\check{\nabla}_\mu \tau_\nu = 0$ and $\check{\nabla}_\rho h^{\mu\nu} = 0$
- has minimal torsion $\check{T}^\rho{}_{\mu\nu} = 2\check{\Gamma}^\rho_{[\mu\nu]} = 2\partial_{[\mu}\tau_{\nu]}$
- zero torsion $\Longleftrightarrow$ absolute time

Associated curvature $\check{R}_{\mu\nu\rho}{}^\sigma$ defined as usual



Newton-Cartan geometry and gravity

Newton-Cartan: clock one-form $\tau_\mu(x^\rho)$ and spatial metric $h^{\mu\nu}(x^\rho)$, connection $\check{\Gamma}^\rho_{\mu\nu}$ and curvature $\check{R}_{\mu\nu\rho}{}^\sigma$

Could also add Bargmann mass field $m_\mu(x^\rho)$
to Newton-Cartan geometry and connection

This allows for a curvature formulation of the Poisson equation

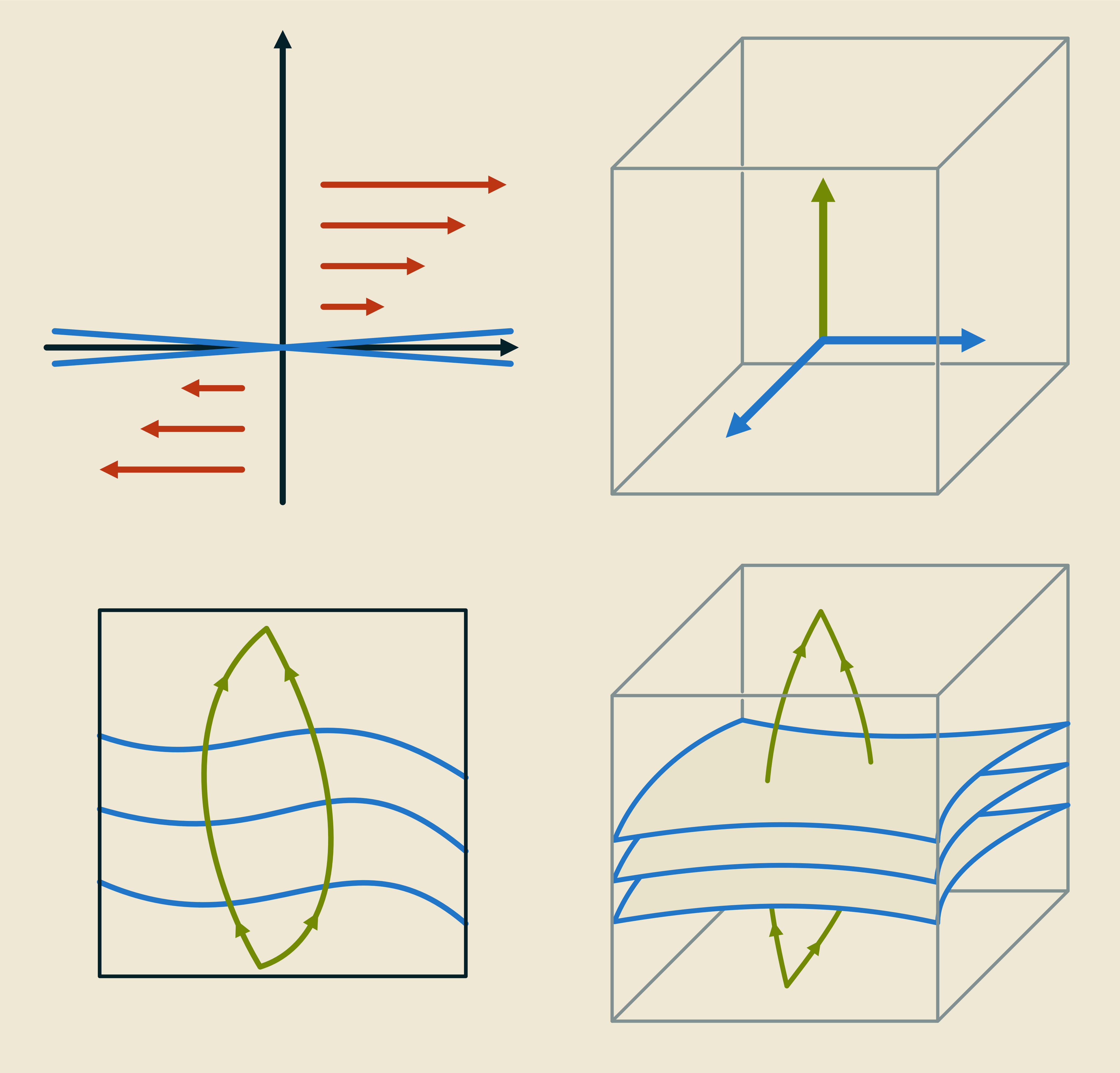
$$\nabla^2\Phi = 4\pi G\rho$$

Using the background geometry

$$\tau_\mu dx^\mu = dt, \qquad h^{\mu\nu}\partial_\mu\partial_\nu = \delta^{ij}\,\partial_i\partial_j\,, \qquad m_\mu dx^\mu = \Phi\, dt$$

the Poisson equation corresponds to

$$\check{R}_{\mu\nu} = 4\pi G\rho\,\tau_\mu\tau_\nu$$



Newton-Cartan geometry and gravity

Newton-Cartan: clock one-form $\tau_\mu(x^\rho)$ and spatial metric $h^{\mu\nu}(x^\rho)$, connection $\check{\Gamma}^\rho_{\mu\nu}$ and curvature $\check{R}_{\mu\nu\rho}{}^\sigma$

Using the background geometry

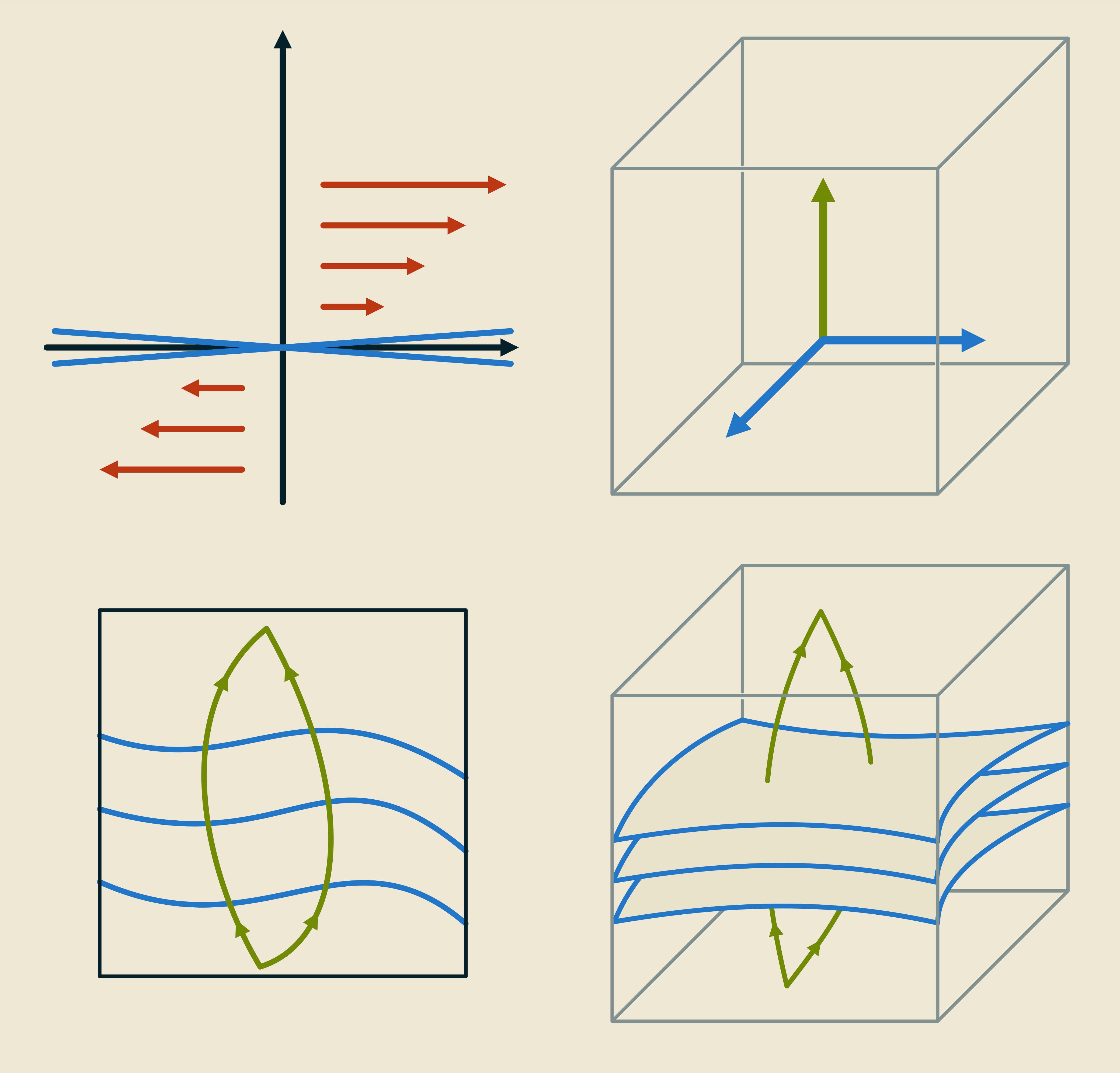
$$\tau_\mu dx^\mu = dt, \qquad h^{\mu\nu}\partial_\mu\partial_\nu = \delta^{ij}\,\partial_i\partial_j\,, \qquad m_\mu dx^\mu = \Phi\, dt$$

get covariant curvature formulation of the Poisson equation

$$\check{R}_{\mu\nu} = 4\pi G\rho\, \tau_\mu\tau_\nu$$

However, this leaves many questions:

- why only this geometry? and why not dynamical?
- where did the m_μ field come from?
- how does Newton-Cartan arise from Lorentzian geometry?
- where did this equation of motion come from?
- geometrically, how does it arise from the Einstein equations?
- what are subleading corrections?



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Newton-Cartan geometry from Lorentzian

From Lorentzian geometry, can get Newton-Cartan by expanding around $c \to \infty$

Two-step process [van den Bleeken] [Hansen, Hartong, Obers]

Rewrite: in a given frame, choose time vector V^μ and rewrite

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}\,, \qquad g^{\mu\nu} = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}$$

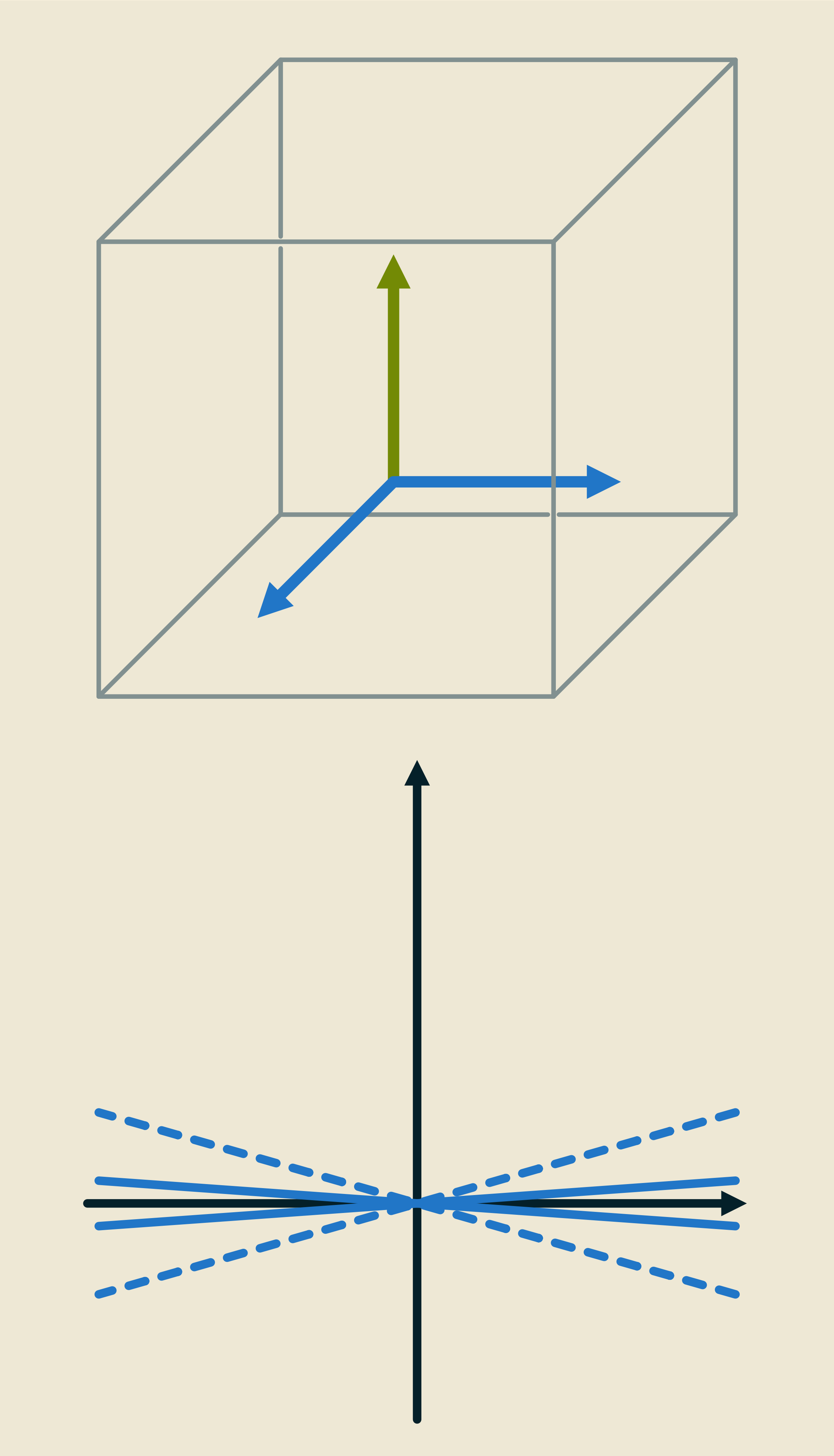
This exposes overall factors of c^2 in the metric

Expand: then Newton-Cartan geometry appears at leading order in $1/c^2$ expansion,

$$T_\mu = \tau_\mu + \frac{1}{c^2} m_\mu + \cdots, \qquad V^\mu = v^\mu + \cdots$$

$$\Pi^{\mu\nu} = h^{\mu\nu} + \frac{1}{c^2}\Phi^{\mu\nu} + \cdots, \qquad \Pi_{\mu\nu} = h_{\mu\nu} + \cdots$$

Local Lorentz symmetry → local Galilei symmetry + corrections



Newton-Cartan geometry from Lorentzian

Newton-Cartan connection $\check{\Gamma}^{\rho}_{\mu\nu}$ and curvature $\check{R}_{\mu\nu\rho}{}^{\sigma}$ can be obtained from Levi-Civita

First, rewrite Levi-Civita to expose overall factors of c^2, which gives

$$\Gamma^{\rho}_{\mu\nu} = c^2 S_{(-2)}{}^{\rho}{}_{\mu\nu} + \bar{C}^{\rho}_{\mu\nu} + S_{(0)}{}^{\rho}{}_{\mu\nu} + \frac{1}{c^2} S_{(2)}{}^{\rho}{}_{\mu\nu} \,,$$

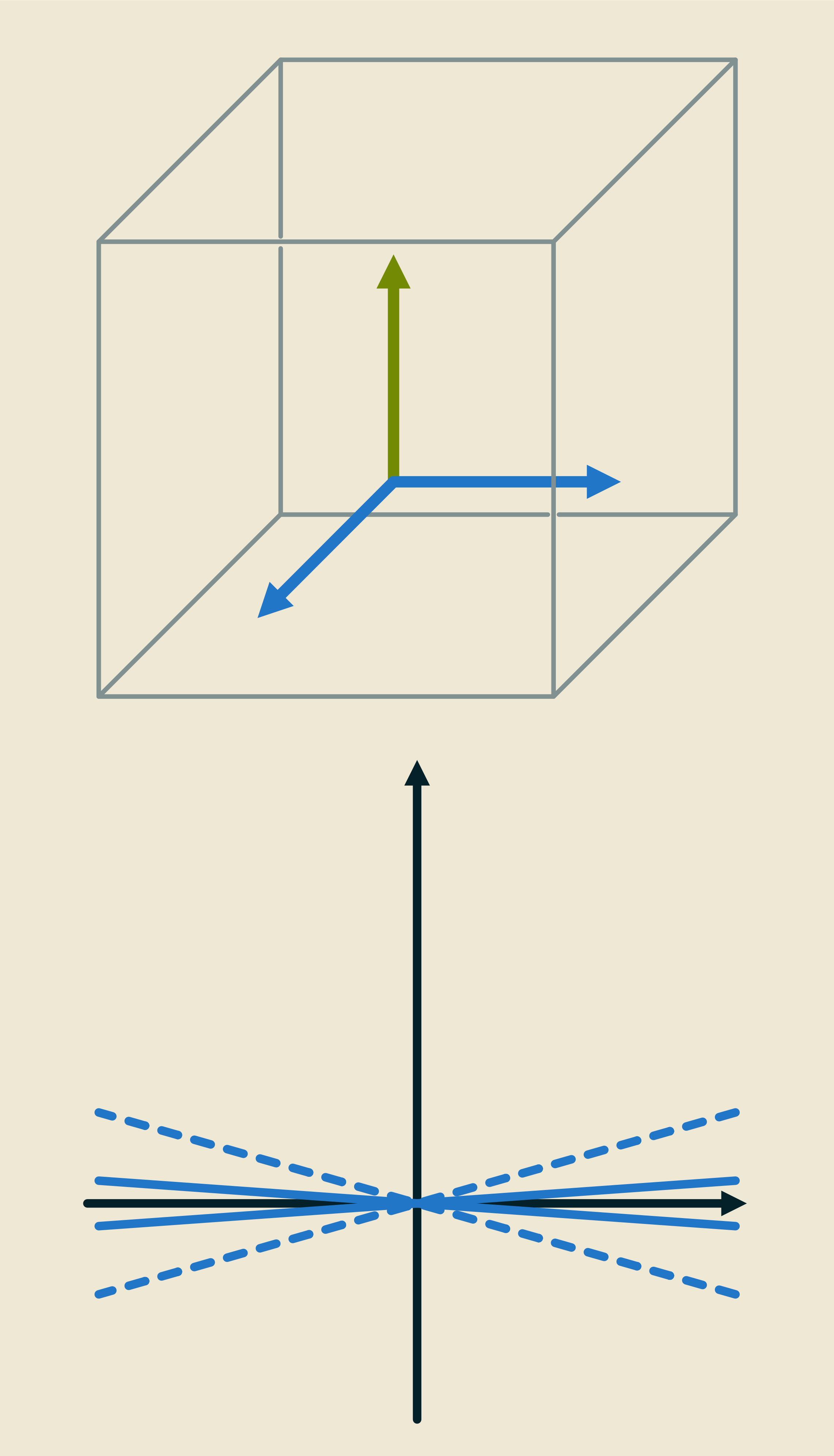
where the $S^{\rho}{}_{\mu\nu}$ are known tensors. Then expand to get $\bar{C}^{\rho}_{\mu\nu} = \check{\Gamma}^{\rho}_{\mu\nu} + \cdots$

Rewrite $\sqrt{-g} = cE$ where $E = \det(T_\mu, \Pi_{\mu\nu})$ and expand $E = e + \cdots$ where $e = \det(\tau_\mu, h_{\mu\nu})$

Finally, we can rewrite Levi-Civita Ricci scalar as

$$R = c^2 \Pi^{\mu\rho}\Pi^{\nu\sigma}\partial_{[\mu}T_{\nu]}\partial_{[\rho}T_{\sigma]} + \Pi^{\mu\nu}\bar{R}_{\mu\nu} + \frac{1}{c^2}\left[\mathscr{K}^{\mu\nu}\mathscr{K}_{\mu\nu} - \mathscr{K}^2\right]$$

where $A_\mu = 2V^\rho \partial_{[\mu}T_{\rho]}$ is acceleration and $\mathscr{K}_{\mu\nu} = -\frac{1}{2}\mathscr{L}_V \Pi_{\mu\nu}$ is extrinsic curvature



Newton-Cartan gravity from GR

Can then rewrite the Einstein-Hilbert action of General Relativity

$$S = \frac{c^3}{16\pi G}\int_M R\sqrt{-g}\,d^dx$$
$$\approx \frac{c^6}{16\pi G}\int_M \left[\Pi^{\mu\rho}\Pi^{\nu\sigma}\partial_{[\mu}T_{\nu]}\partial_{[\rho}T_{\sigma]} + \frac{1}{c^2}\Pi^{\mu\nu}\bar{R}_{\mu\nu} + \frac{1}{c^4}\left(\mathscr{K}^{\mu\nu}\mathscr{K}_{\mu\nu} - \mathscr{K}^2\right)\right]E\,d^dx$$

From Lorentzian point of view, this is a somewhat strange thing to do!

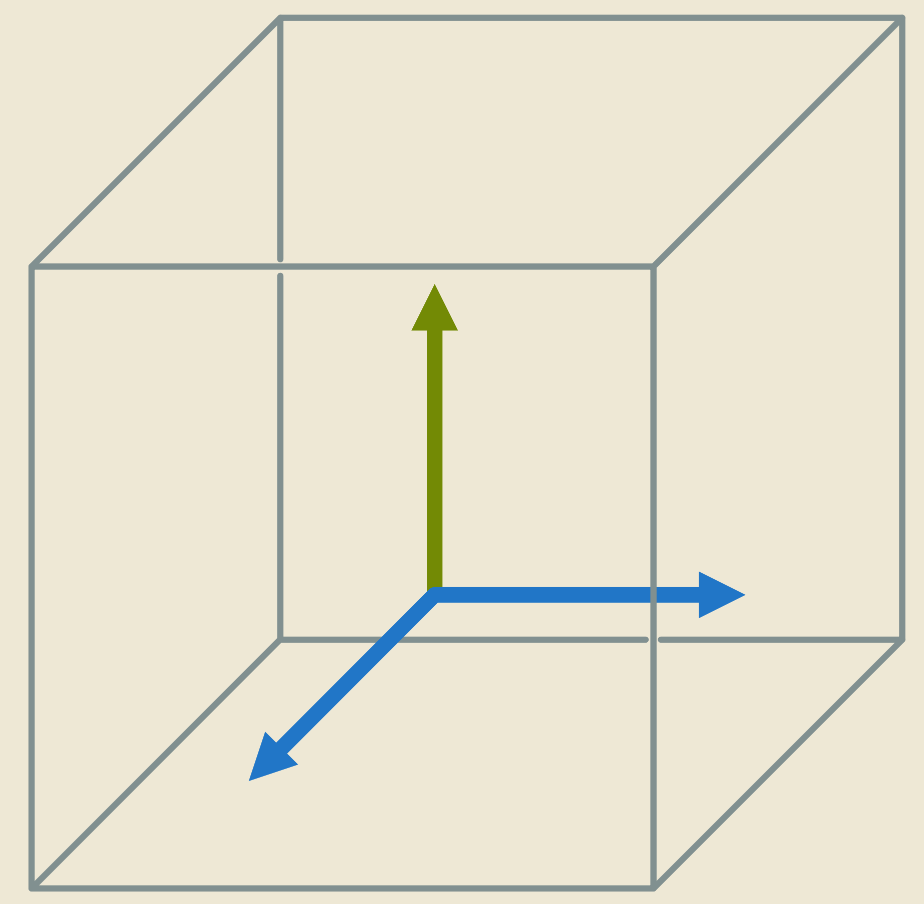
($\bar{C}^\rho_{\mu\nu} = \check{\Gamma}^\rho_{\mu\nu} + \cdots$ is neither flat nor Lorentz-metric-compatible nor torsion-free)

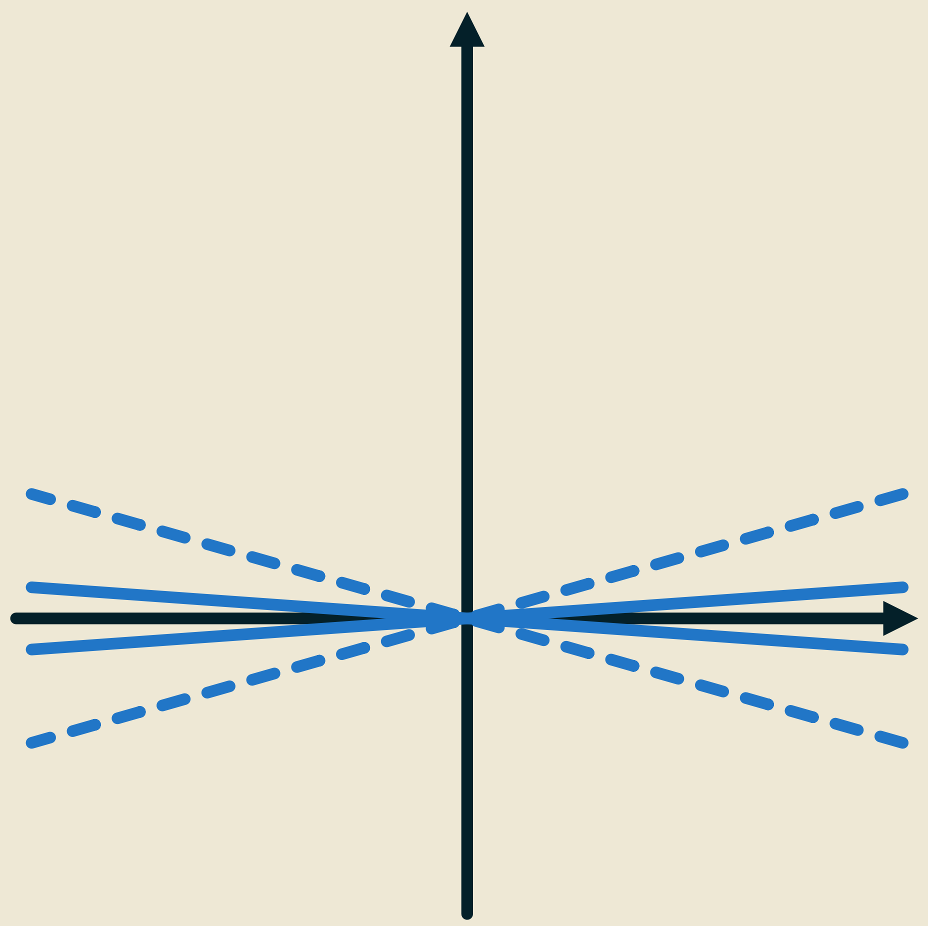
However it allows us to expand the action in $1/c^2$, *non-relativistic geometric expansion!*

$$S = c^6 S_{\mathrm{LO}} + c^4 S_{\mathrm{NLO}} + c^2 S_{\mathrm{NNLO}} + \cdots$$

At leading order,

$$S_{\mathrm{LO}} = \frac{1}{16\pi G}\int h^{\mu\rho}h^{\nu\sigma}\partial_{[\mu}\tau_{\nu]}\partial_{[\rho}\tau_{\sigma]}\,e\,d^dx \quad \text{leads to EOM} \quad \tau\wedge d\tau = 0$$





Newton-Cartan gravity from GR

First rewrite the Einstein-Hilbert action of General Relativity

$$S = \frac{c^3}{16\pi G}\int_M R\sqrt{-g}\,d^dx$$
$$\approx \frac{c^6}{16\pi G}\int_M \left[\Pi^{\mu\rho}\Pi^{\nu\sigma}\partial_{[\mu}T_{\nu]}\partial_{[\rho}T_{\sigma]} + \frac{1}{c^2}\Pi^{\mu\nu}\bar{R}_{\mu\nu} + \frac{1}{c^4}\left(\mathscr{K}^{\mu\nu}\mathscr{K}_{\mu\nu} - \mathscr{K}^2\right)\right] E\,d^dx$$

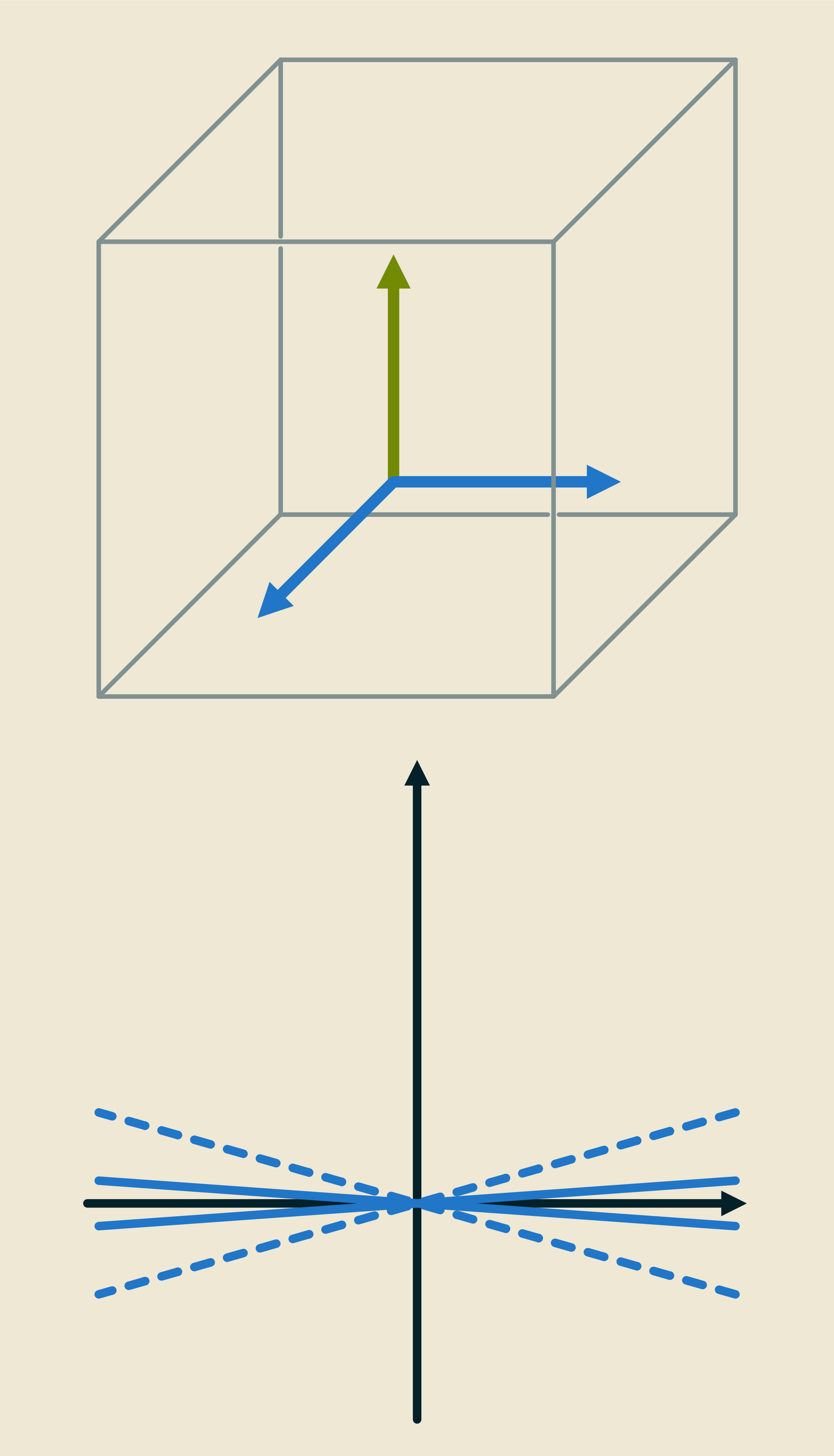
then expand the action in powers of $1/c^2$

$$S = c^6 S_{\text{LO}} + c^4 S_{\text{NLO}} + c^2 S_{\text{NNLO}} + \cdots$$

At leading order, get foliation condition

$$S_{\text{LO}} = \frac{1}{16\pi G}\int h^{\mu\rho}h^{\nu\sigma}\partial_{[\mu}\tau_{\nu]}\partial_{[\rho}\tau_{\sigma]}\,e\,d^dx \quad \text{leads to EOM} \quad \tau\wedge d\tau = 0$$

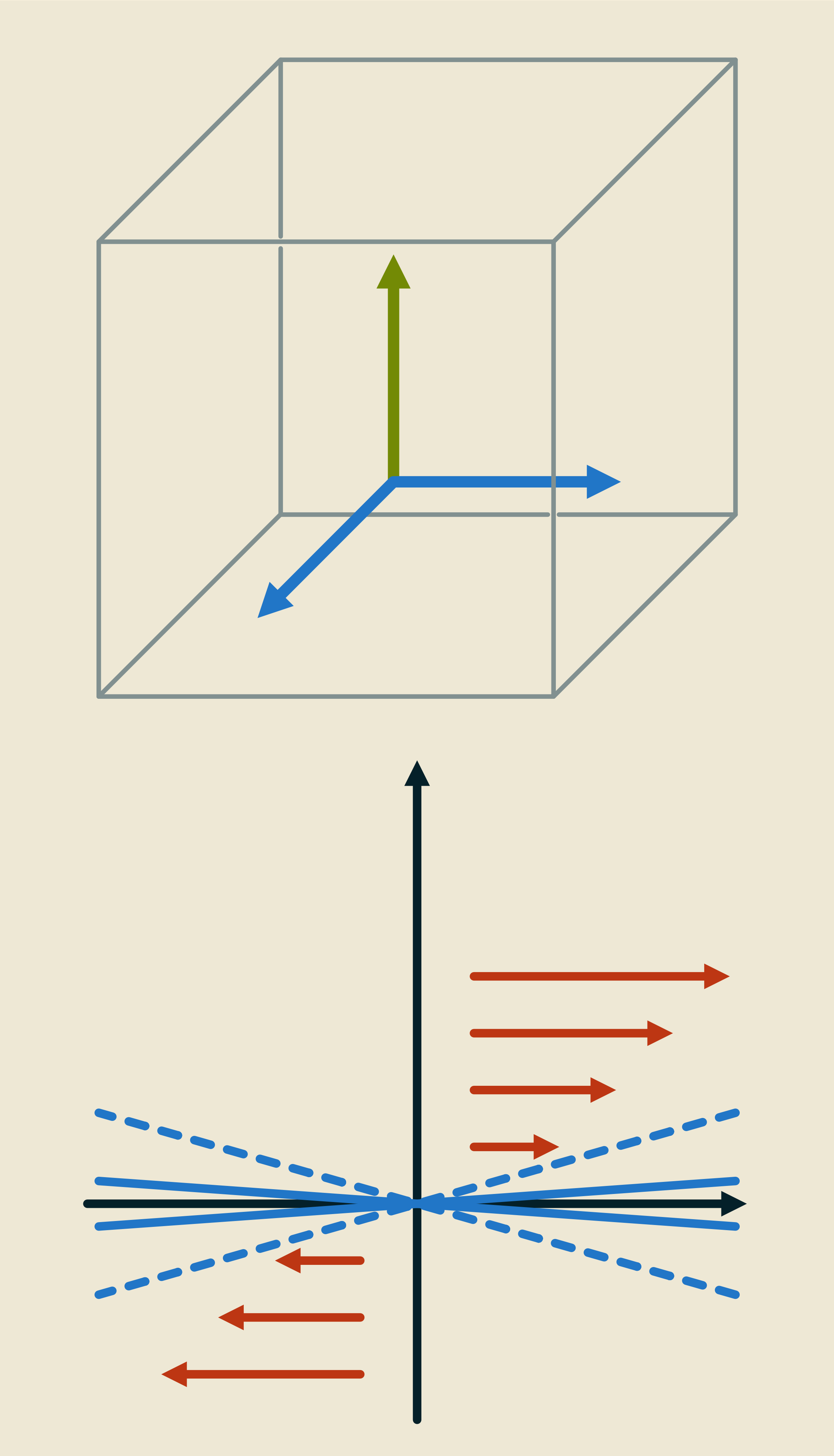
At next-to-next-to-leading order *retrieve Poisson equation* as subset of EOM!
(But pretty complicated!)



Newton-Cartan summary

So far,

- introduced Newton-Cartan geometry with local Galilean boosts
- obtained it at leading order in $c \to \infty$ expansion of Lorentzian geometry
- applied to the Einstein-Hilbert action
- derivation of Poisson equation from *action principle for dynamical geometry!*



Did not cover expansion of matter fields, solutions, geodesics …
[Van den Bleeken] [Hansen, Hartong, Obers]

Open problems:

- establish well-posed initial value problem
- NNLO theory looks complicated, but is it still easier than GR?
- make contact with numerical simulations?

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Carroll symmetries

Are used to `relativistic' Lorentz boosts

$$t \to t + \beta x, \qquad x \to x + \beta t$$

Non-relativistic limit $c \to \infty$ gives Galilean boosts

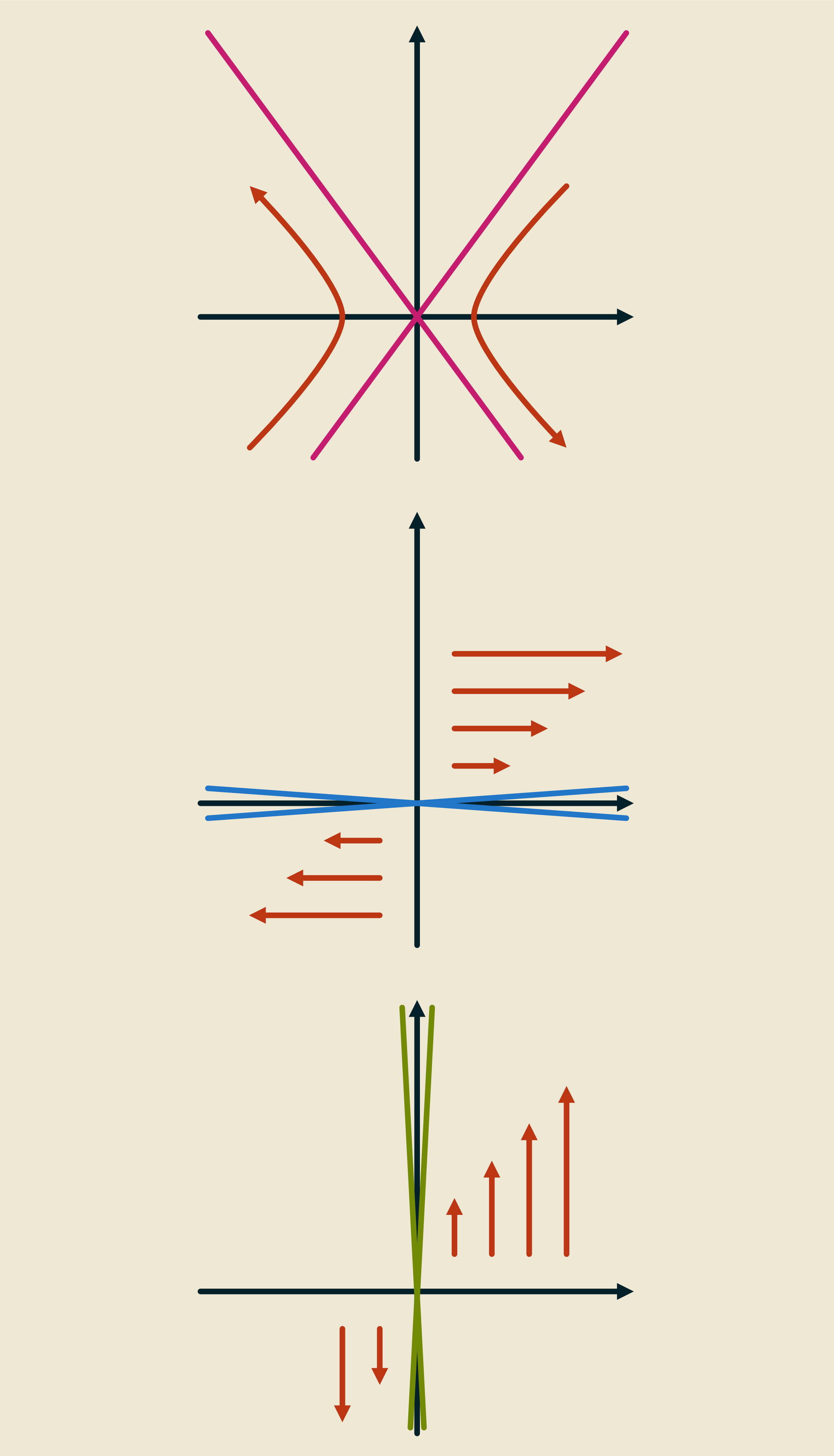
$$t \to t, \qquad x \to x + vt$$

Instead, taking $c \to 0$ gives Carroll boosts [Levy-Leblond] [Sen Gupta]

$$t \to t + \lambda x \qquad x \to x$$

Less obviously physical, but

- ultra-local behavior leads to solvable systems
- appears in Lorentzian geometry on null surfaces such as $\mathscr{I}^+$
- BMS asymptotic symmetries are isomorphic to conformal Carroll algebra [Duval, Gibbons, Horvathy, Zhang]



Carroll geometry

Are used to `relativistic' Lorentz boosts

$$t \to t + \beta x, \qquad x \to x + \beta t$$

Ultra-local Carroll limit $c \to \infty$ gives Carroll boosts

$$t \to t + \lambda x\,, \quad x \to x \quad \text{and} \quad \partial_t \to \partial_t\,, \quad \partial_x \to \partial_x + \lambda \partial_t$$

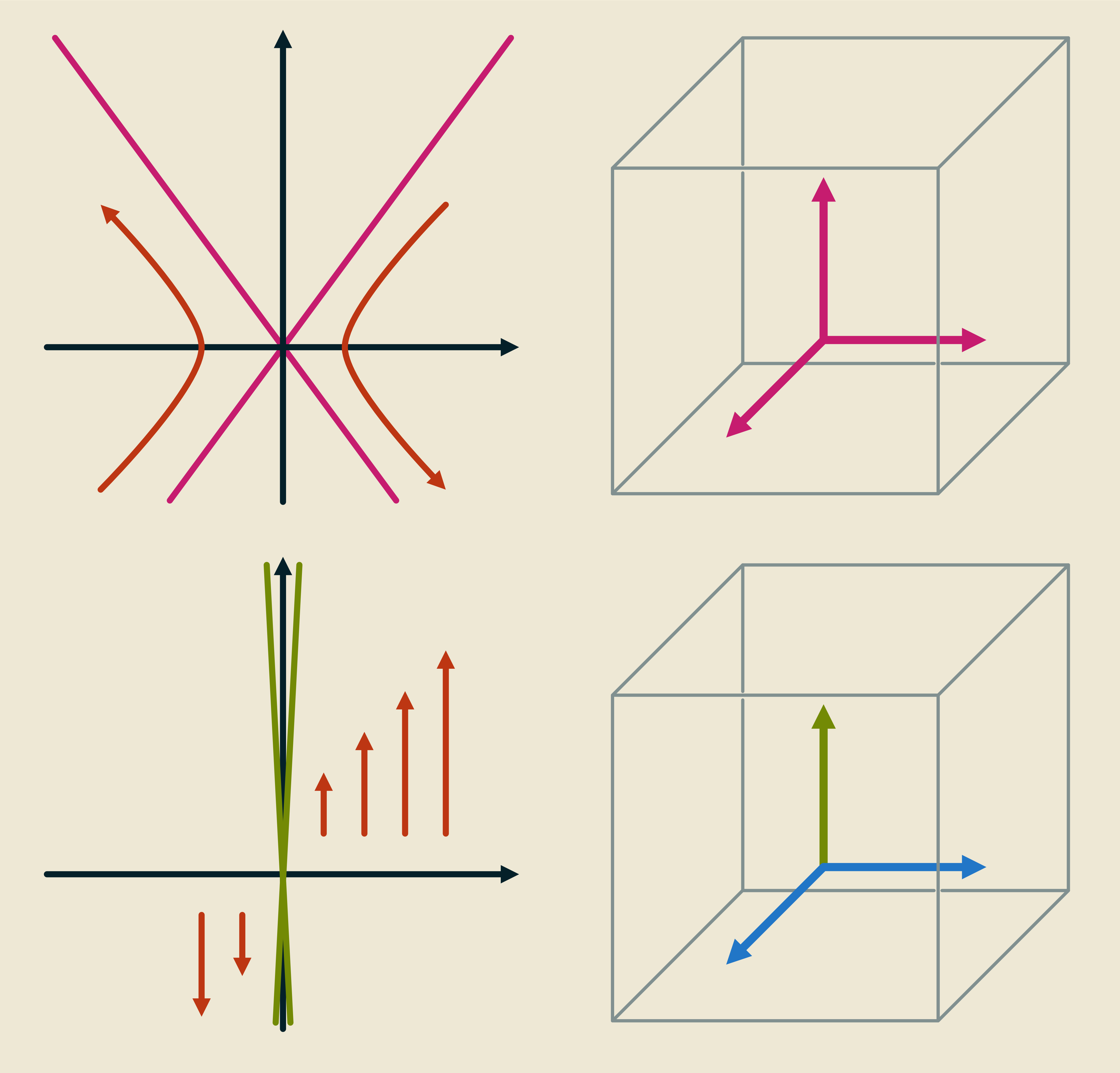
Geometry from time vector field $v^\mu(x^\rho)$ and spatial metric $h_{\mu\nu}(x^\rho)$
[Duval, Gibbons, Horvathy, Zhang] [Hartong] [Ciambelli, Marteau, Petropoulos…]
[Hansen, Obers, GO, Søgaard] …

Complement with inverse $\tau_\mu(x^\rho)$ and $h^{\mu\nu}(x^\rho)$, satisfy

$$v^\mu h_{\mu\nu} = 0, \quad \tau_\mu h^{\mu\nu} = 0, \quad v^\mu \tau_\mu = -1, \quad \delta^\mu_\nu = -v^\mu \tau_\nu + h^{\mu\rho} h_{\rho\nu}$$

Transform under local Carroll boosts $\lambda_\mu(x^\rho)$ as

$$\delta_\lambda \tau_\mu = \lambda_\mu\,, \qquad \delta_\lambda h^{\mu\nu} = \lambda^\mu v^\nu + v^\mu \lambda^\nu$$



Carroll symmetries and flat holography

Holographic dual field theory for asymptotically flat spacetimes?

In 3+1 dim: BMS_4 asymptotic symmetries on $\mathscr{I}^+ \simeq \mathbb{R} \times S^2$

supertranslations $u \to u + f(z, \bar{z})$

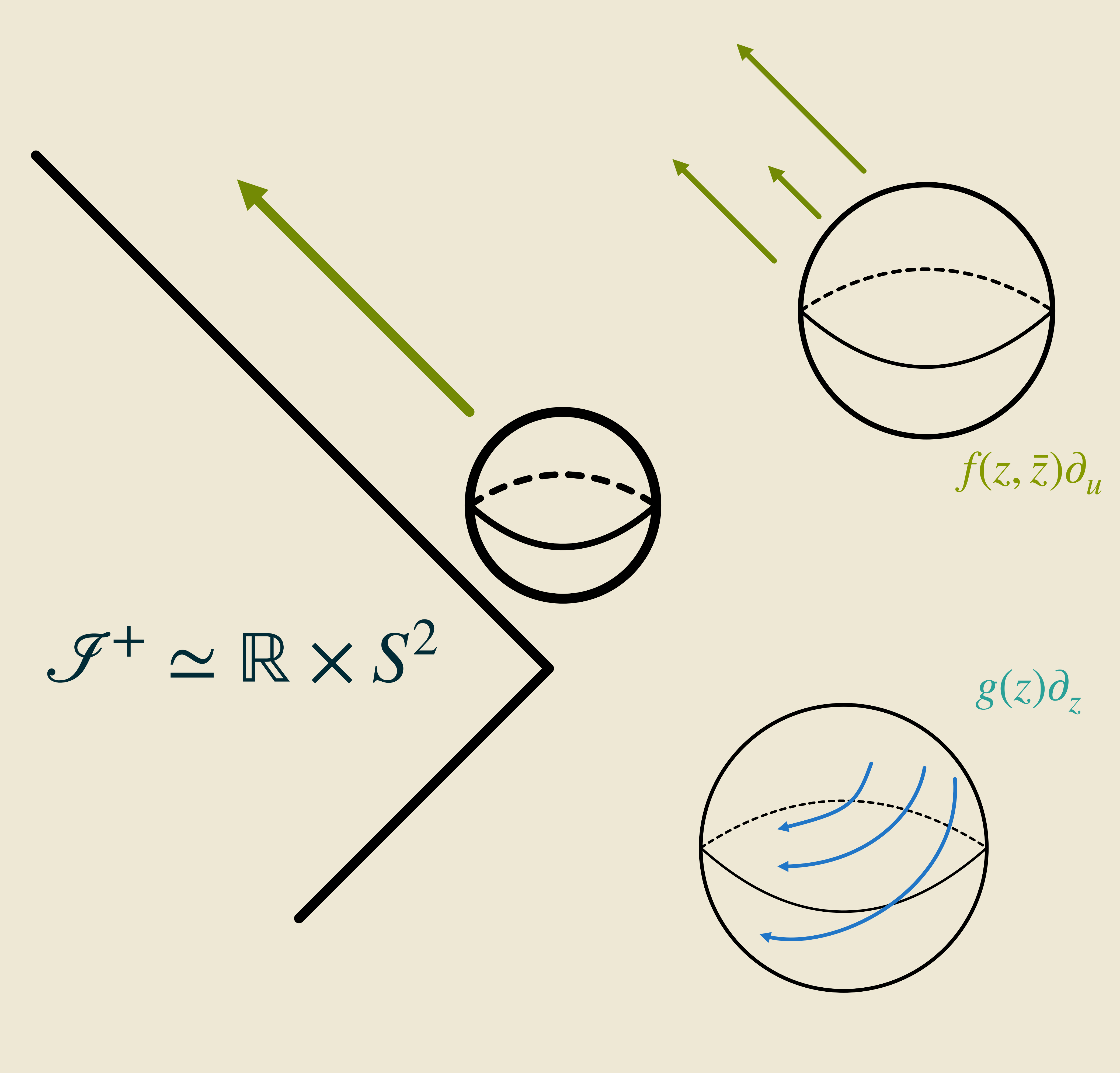
- $\sim$ Carroll boosts at each $(z, \bar{z})$
- suggests 3d Carrollian CFT dual: $\mathrm{BMS}_4 \simeq \mathrm{CCar}_3$

superrotations $z \to g(z), \quad \bar{z} \to \bar{g}(\bar{z})$

- Virasoro symmetries of CFT_2
- suggests 2d celestial CFT dual: CCFT_2

u-direction enters in CCFT_2 as conformal weight $\Delta \in 1 + i\mathbb{R}$
[Pasterski, Shao, Strominger]

Few explicit CCFT_2 theories known,
but *can construct CCar_3 examples from $c \to 0$ limit*



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Carroll expansion of GR

Can also apply $c \to 0$ Carroll expansion to general relativity [Hansen, Obers, GO, Søgaard]

Using previous metric parametrization and expansion

$$g^{\mu\nu} = -\frac{1}{c^2}V^\mu V^\nu + \Pi^{\mu\nu} \qquad V^\mu = v^\mu + c^2 M^\mu + \cdots \qquad \Pi^{\mu\nu} = h^{\mu\nu} + \cdots$$

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu} \qquad T_\mu = \tau_\mu + \cdots \qquad \Pi^{\mu\nu} = h^{\mu\nu} + \cdots$$

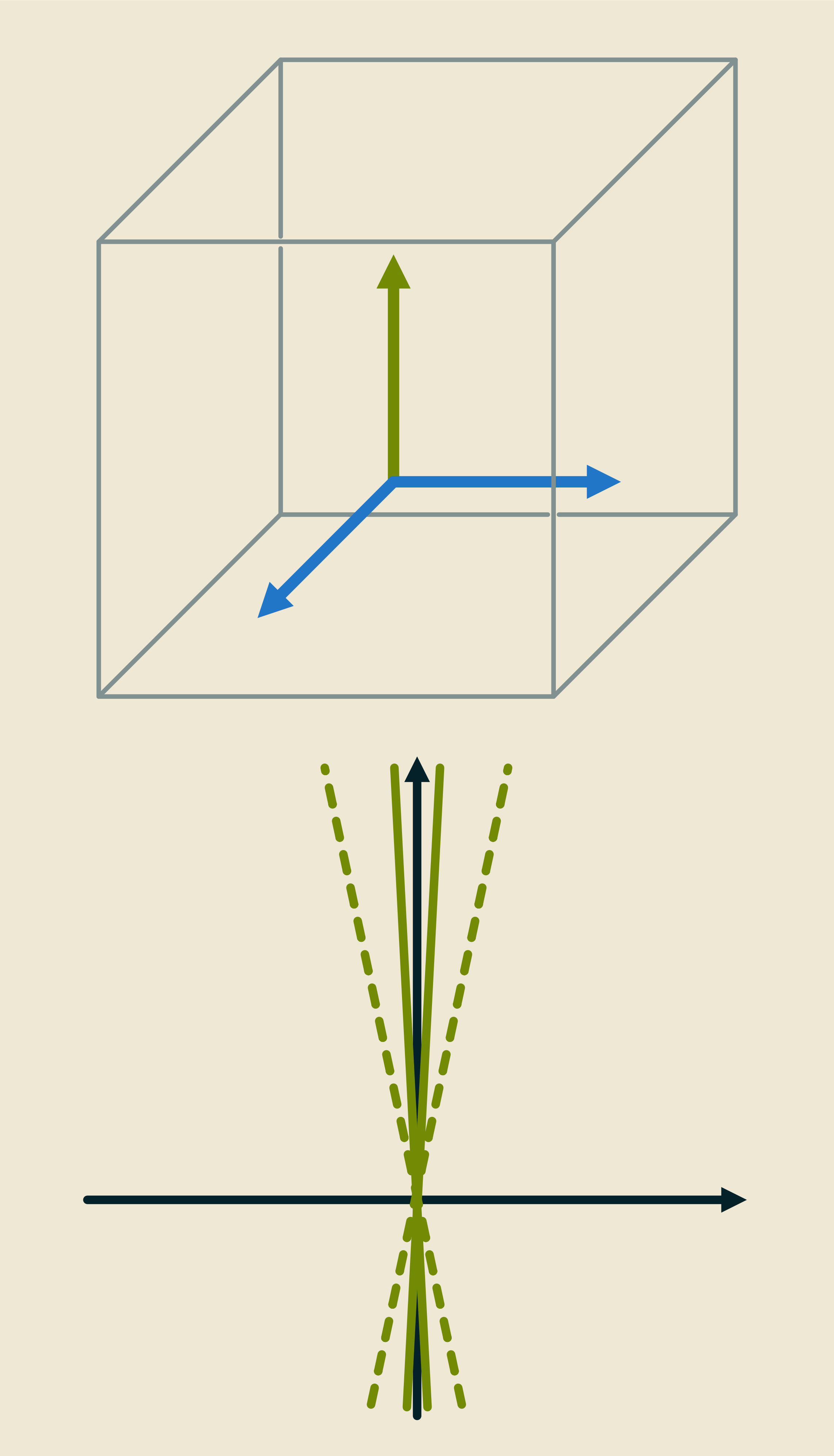
can rewrite Einstein—Hilbert action as

$$S = \frac{c^3}{16\pi G}\int_M R\sqrt{-g}\, d^d x$$

$$\approx \frac{c^2}{16\pi G}\int_M \left[\left(\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - \mathcal{K}^2\right) + c^2\Pi^{\mu\nu}\tilde{R}_{\mu\nu} + \frac{c^4}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}(dT)_{\mu\rho}(dT)_{\nu\sigma}\right] E\, d^d x$$

To leading order, this gives the timelike (or electric) Carroll gravity action

$$S = = \frac{1}{16\pi G}\int_M \left[K^{\mu\nu}K_{\mu\nu} - K^2\right] e\, d^d x,$$



Carroll expansion of GR: timelike

Leading-order $c \to 0$ expansion of GR gives timelike/electric Carroll gravity action

$$S = \frac{1}{16\pi G} \int_M \left[K^{\mu\nu} K_{\mu\nu} - K^2 \right] e \, d^d x,$$

Agrees with Hamiltonian limits [Henneaux, Salgado-Rebodello]

EOM split into constraint and evolution equations [Hansen, Obers, GO, Søgaard] [Dautourt]

$$0 = K^{\mu\nu} K_{\mu\nu} - K^2$$

$$0 = h^{\rho\sigma} \tilde{\nabla}_\rho (K_{\sigma\mu} - K h_{\sigma\mu})$$

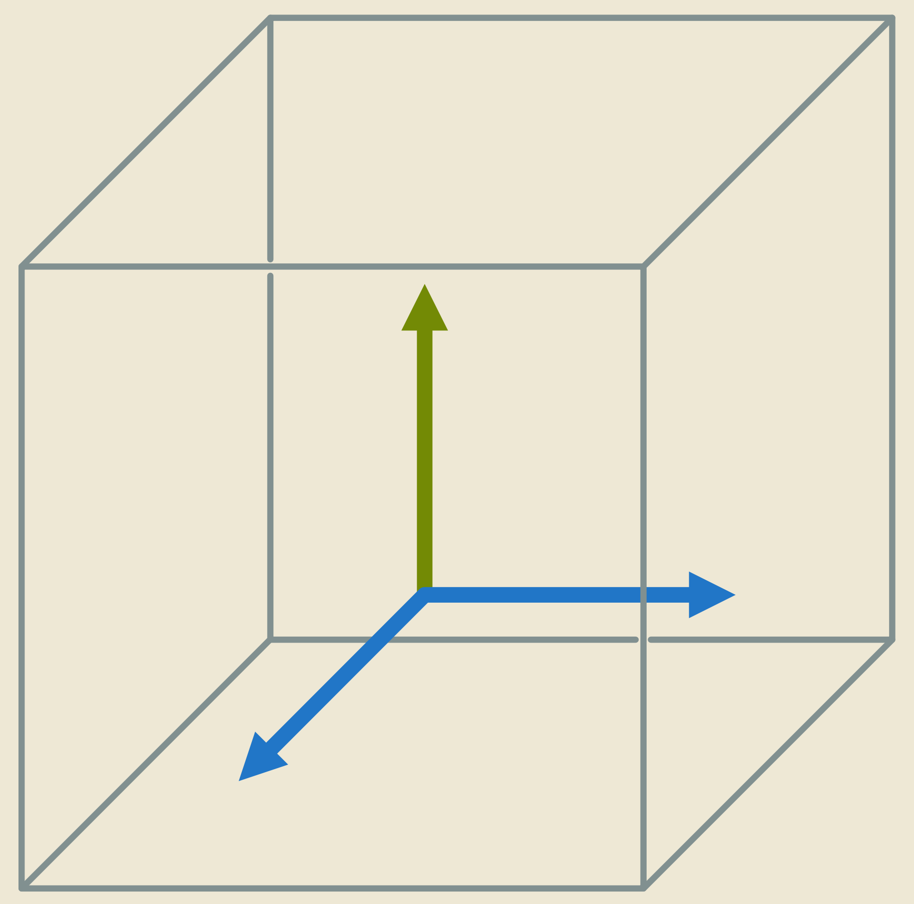
$$\mathscr{L}_v K_{\mu\nu} = -2 K_\mu{}^\rho K_{\rho\nu} + K K_{\mu\nu}$$

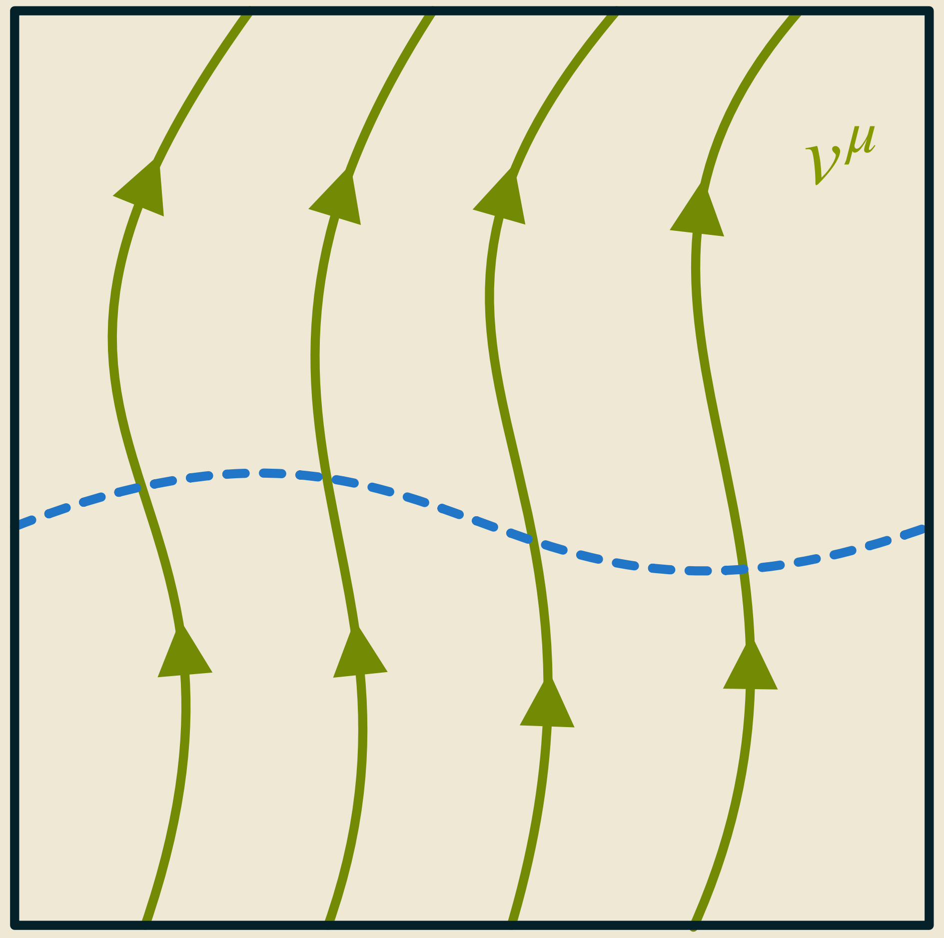
Limit of 3+1 Lorentzian EOM. Remarkably, evolution can be *solved analytically!*

$\implies$ simpler also at NLO?

Found constraint solutions with physical angular and linear momentum

$\implies$ solvable subleading dynamics? Relation to BKL limit?





Carroll expansion of GR: spacelike

From other limit can get spacelike/magnetic Carroll gravity action

$$S = \frac{1}{16\pi G}\int_M \left[h^{\mu\nu}\tilde{R}_{\mu\nu} + \chi^{\mu\nu}K_{\mu\nu}\right] e\, d^d x,$$

Subset of full NLO action in $c \to 0$ expansion, no dynamics since $K_{\mu\nu} \sim \mathscr{L}_v h_{\mu\nu} = 0$

Projecting EOM on spatial hypersurface, constraint is now

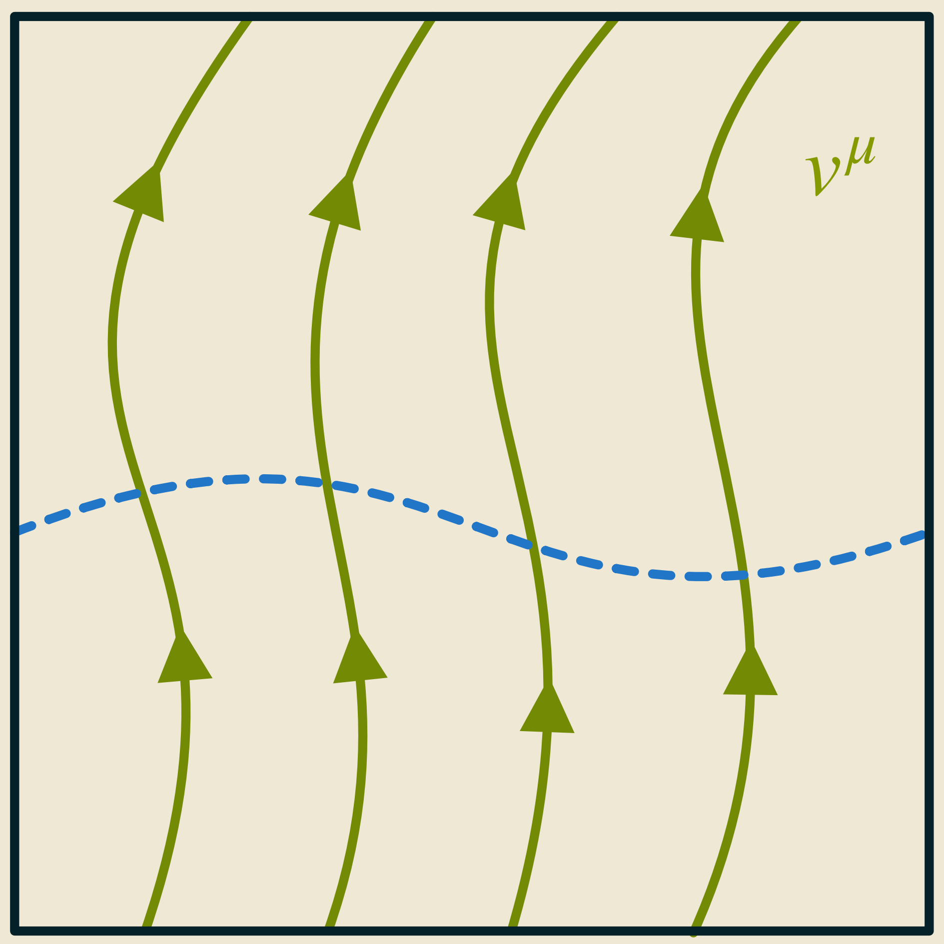
$$0 = h^{\mu\nu}\hat{R}_{\mu\nu}$$

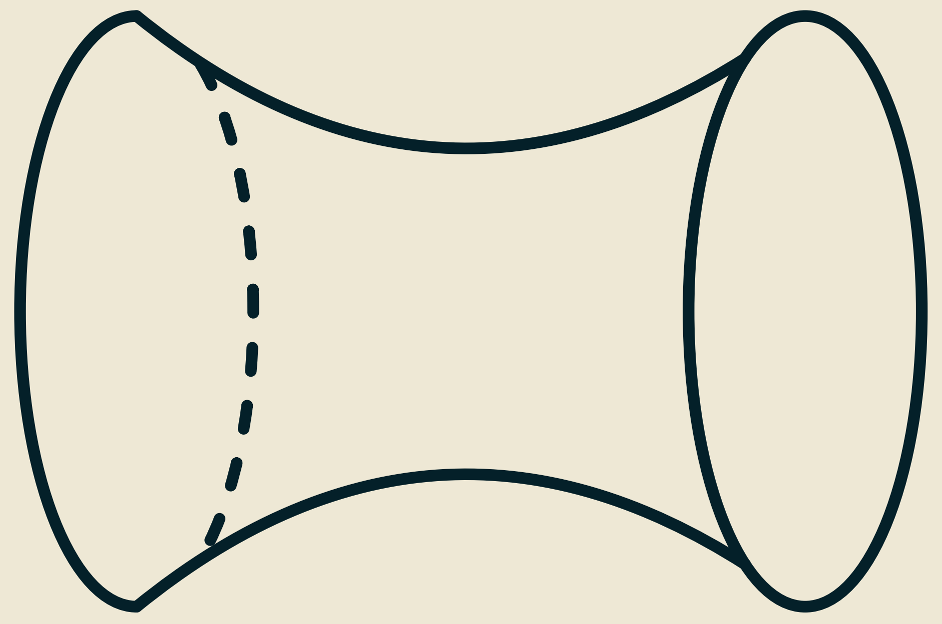
In 3+1 Lorentzian EOM this is responsible for massive solutions $\sim -1 + \dfrac{2GM}{r}$

Indeed now find isotropic 'black hole' solution

$$v^\mu \partial_\mu = \frac{M+2r}{M-2r}\partial_t \qquad h_{\mu\nu}dx^\mu dx^\nu = \left(1 + \frac{M}{2r}\right)^4 \delta_{ij}\, dx^i dx^j$$

Dynamics in full NLO theory?





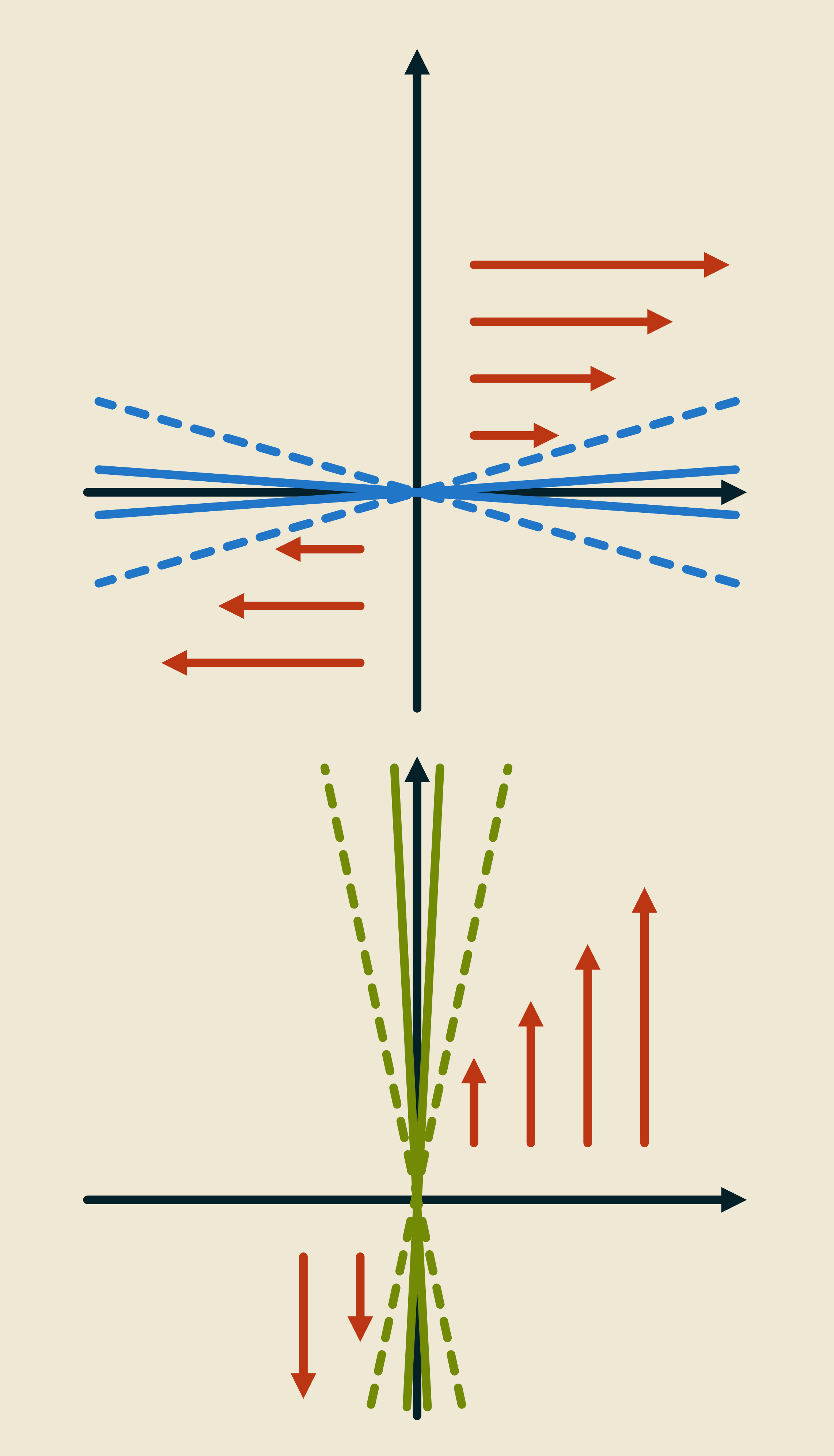
Outlook

Wrapping up, we have

- introduced Newton-Cartan geometry with local Galilean boosts
- obtained it from $c \to \infty$ expansion of GR
- sketched derivation of Poisson equation from action principle

and then

- introduced Carroll geometry with local Carroll boosts
- obtained it from $c \to \infty$ expansion of GR
- found rich and analytically solvable equations at leading order



Open problems:

- how far can analytic control extend to subleading orders?
- make contact with post-Newtonian and numerical simulations?