Conformal Carroll Scalar Actions

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Based mainly on 2207.03468 with Stefano Baiguera, Watse Sybesma and Benjamin Søgaard

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Carroll limits and flat holography

Are used to 'relativistic' Lorentz boosts

$$t \to t + \beta x, \qquad x \to x + \beta t$$

Taking $c \to 0$ limit gives Carroll boosts [Levy-Leblond] [Sen Gupta]

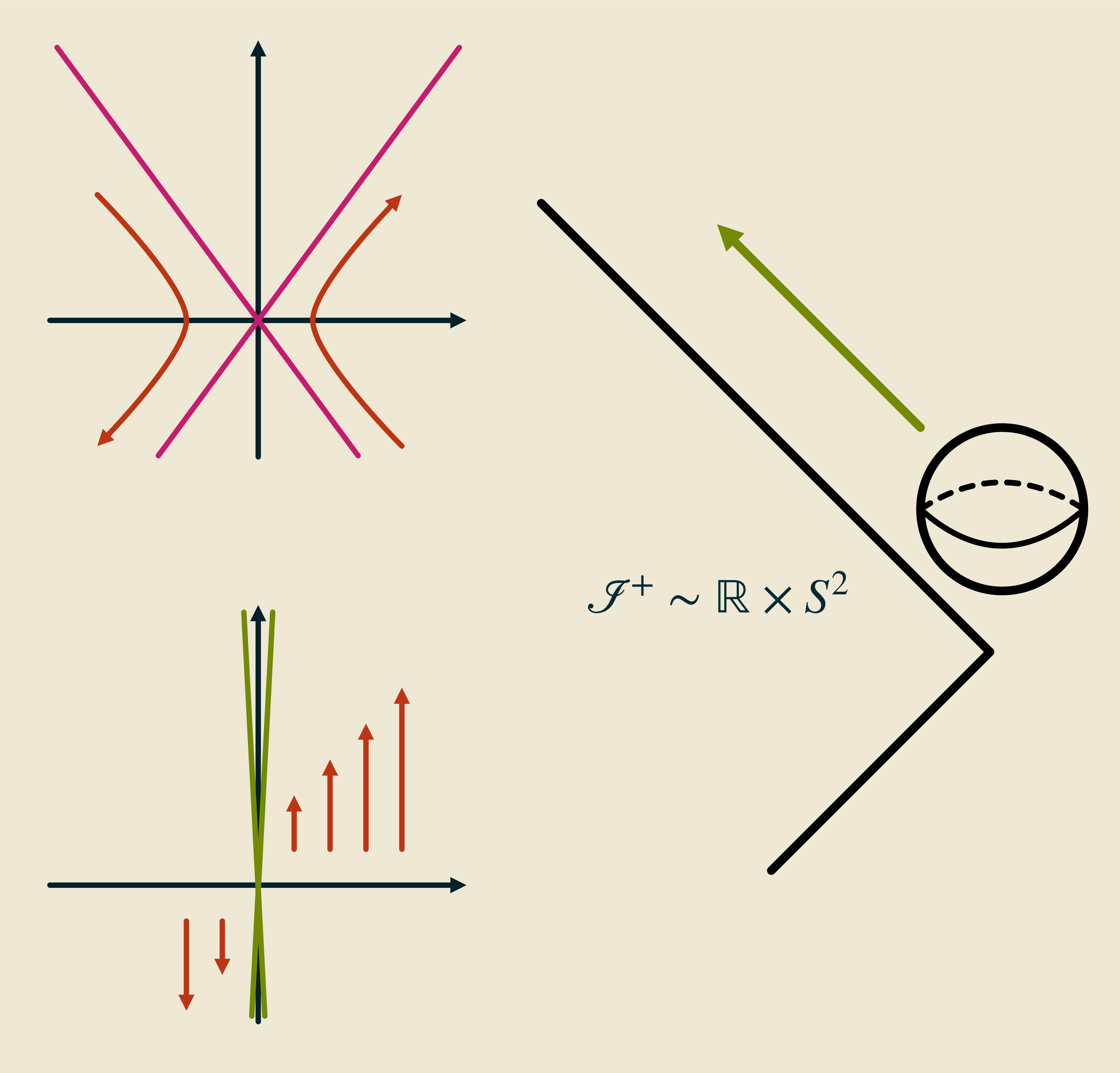
$$t \to t + \lambda x \qquad x \to x$$

Not obviously physical, but:

- ultra-local behavior leads to solvable systems such as integrable BKL-type dynamics in GR [see Niels' talk]
- BMS = conformal Carroll algebra at $\mathscr{I}^+$ [Duval, Gibbons, Horvathy, Zhang]
- Flat space holography, relation to celestial approach [Donnay, Fiorucci, Herfray, Ruzziconi] [Bagchi, Banerjee, Basu, Dutta]

Expand Lorentz-invariant actions to get Carroll-invariant actions

$\Longrightarrow$ use this to *construct explicit flat space dual field theories* for example from limits of top-down AdS/CFT settings?



Main goal: find explicit actions for conformal Carroll theories

- Obtain Carroll geometry from $c \to 0$ expansion of Lorentzian
- Discuss Carroll boost symmetries and their consequences
- Use expansion to construct conformal Carroll action from

$$S = -\frac{1}{2}\int d^d x \sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{d-2}{4(d-1)}R\phi^2\right)$$

Lorentzian review

Local Lorentz transformations act on vielbeine ${E^A}_\mu$ and inverse ${\Theta_A}^\mu$ as

$$\delta_\Lambda {E^A}_\mu = {\Lambda^A}_B {E^B}_\mu \quad , \quad \delta_\Lambda {\Theta_A}^\mu = -{\Lambda^B}_A {\Theta_B}^\mu$$

Define energy-momentum tensor using vielbein variation

$${T^\mu}_A = \frac{1}{E}\frac{\delta S}{\delta {E^A}_\mu}$$

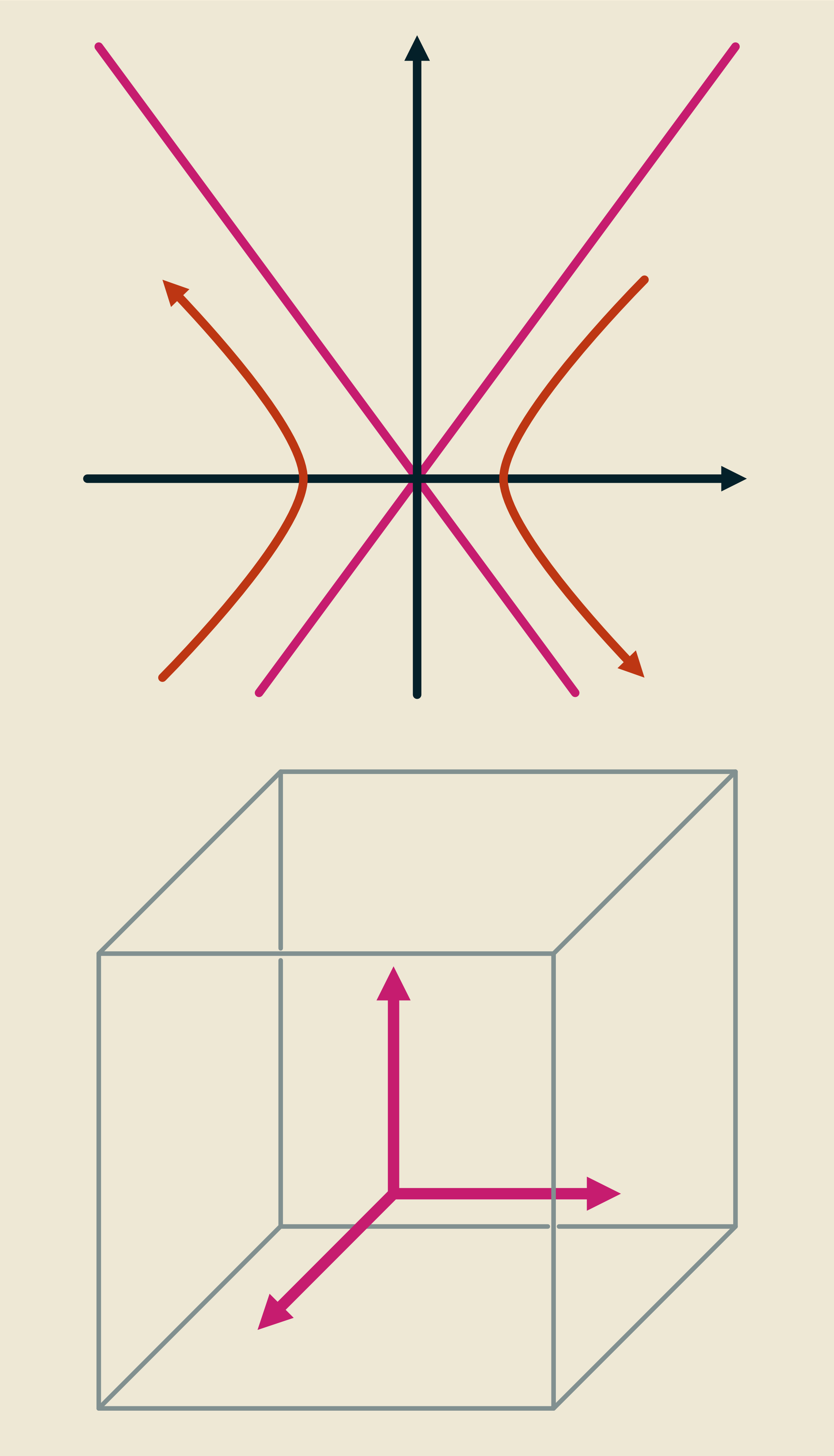
Then Ward identity for local Lorentz symmetries is

$$0 = \delta_\Lambda S = \int d^d x E \left({T^\mu}_A \delta_\Lambda {E^A}_\mu \right) \quad \Longrightarrow \quad 0 = {T^\mu}_A {\Lambda^A}_B {E^B}_\mu = T^{AB}\Lambda_{AB}$$

so energy-momentum tensor is symmetric

Likewise, Weyl symmetries $\delta_\Omega {E^A}_\mu = \Omega {E^A}_\mu$ imply it is traceless, ${T^\mu}_\mu = 0$

Both hold for conformal scalar $S = -\frac{1}{2}\int d^d x \sqrt{-g} \left(g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi + \frac{d-2}{4(d-1)} R\phi^2 \right)$



Carroll geometry from expansion

To do ultra-local $c \to 0$ expansion, first split off factors of c in vielbeine

$$E^A{}_\mu = \left(cT_\mu, E^a{}_\mu\right) \,, \qquad \Theta_A{}^\mu = \left(-\frac{1}{c}V^\mu, \Theta_a{}^\mu\right)$$

Under Lorentz transformations $\Lambda^0{}_a$ these variables transform as

$$\delta_\Lambda T_\mu = \frac{1}{c}\Lambda^0{}_a E^a{}_\mu \qquad \delta_\Lambda V^\mu = c\Lambda^a{}_0 T_\mu$$

$$\delta_\Lambda E^a{}_\mu = c\Lambda^a{}_0 T_\mu \qquad \delta_\Lambda \Theta_a{}^\mu = \frac{1}{c}\Lambda^0{}_a V^\mu$$

Then expand these variables around $c \to 0$ as

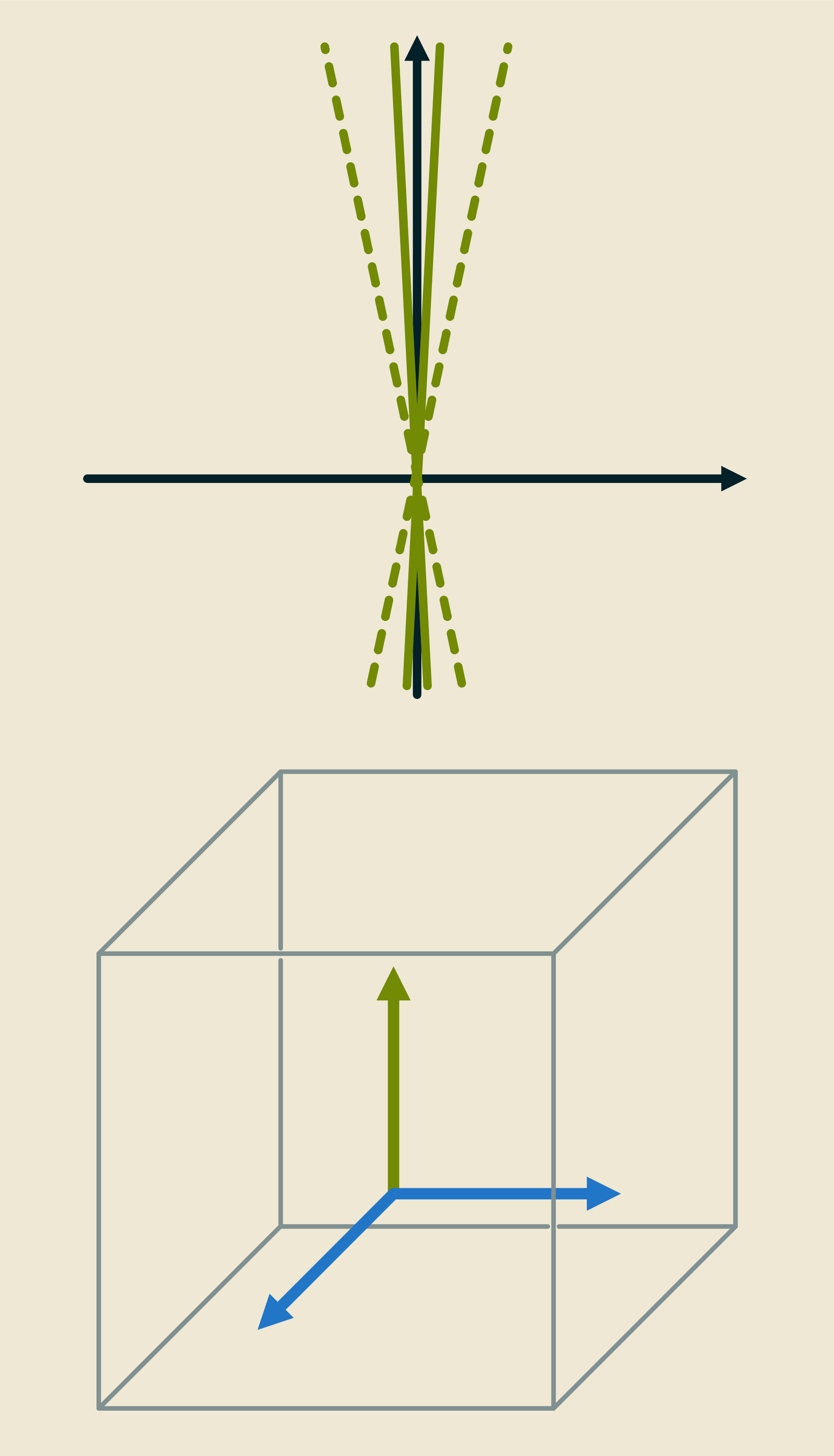
$$T_\mu = \tau_\mu + \mathcal{O}(c^2) \qquad V^\mu = v^\mu + \mathcal{O}(c^2)$$

$$E^a{}_\mu = e^a{}_\mu + \mathcal{O}(c^2) \qquad \Theta_a{}^\mu = \theta^a{}_\mu + \mathcal{O}(c^2)$$

Limit of Lorentz boosts gives local Carroll boosts with $\Lambda^0{}_a = c\lambda_a + \cdots$

$$\delta_\lambda \tau_\mu = \lambda_a e^a{}_\mu \qquad \delta_\lambda v^\mu = 0$$

$$\delta_\lambda e^a{}_\mu = 0 \qquad \delta_\lambda \theta_a{}^\mu = \lambda_a v^\mu$$



In the $c \to 0$ limit, local Carroll boosts *follow inevitably from local Lorentz boosts!*

Carroll boosts and energy flux

Fundamental Carroll data is time vector field v^μ and spatial metric $h_{\mu\nu} = \delta_{ab} e^a{}_\mu e^b{}_\nu$

$\implies$ fiber bundle with vertical v^μ and base metric $h_{\mu\nu}$

Can add inverse τ_μ ~ Ehresmann connection and $h^{\mu\nu}$ ~ horizontal projector $h^{\mu\rho} h_{\rho\nu}$

These transform under Carroll boosts $\lambda_\mu = e^a{}_\mu \lambda_a$

$$\delta_\lambda \tau_\mu = \lambda_\mu \,, \qquad \delta_\lambda h^{\mu\nu} = \lambda^\mu v^\nu + v^\mu \lambda^\nu \qquad \delta_\lambda v^\mu = 0 \,, \qquad \delta_\lambda h_{\mu\nu} = 0$$

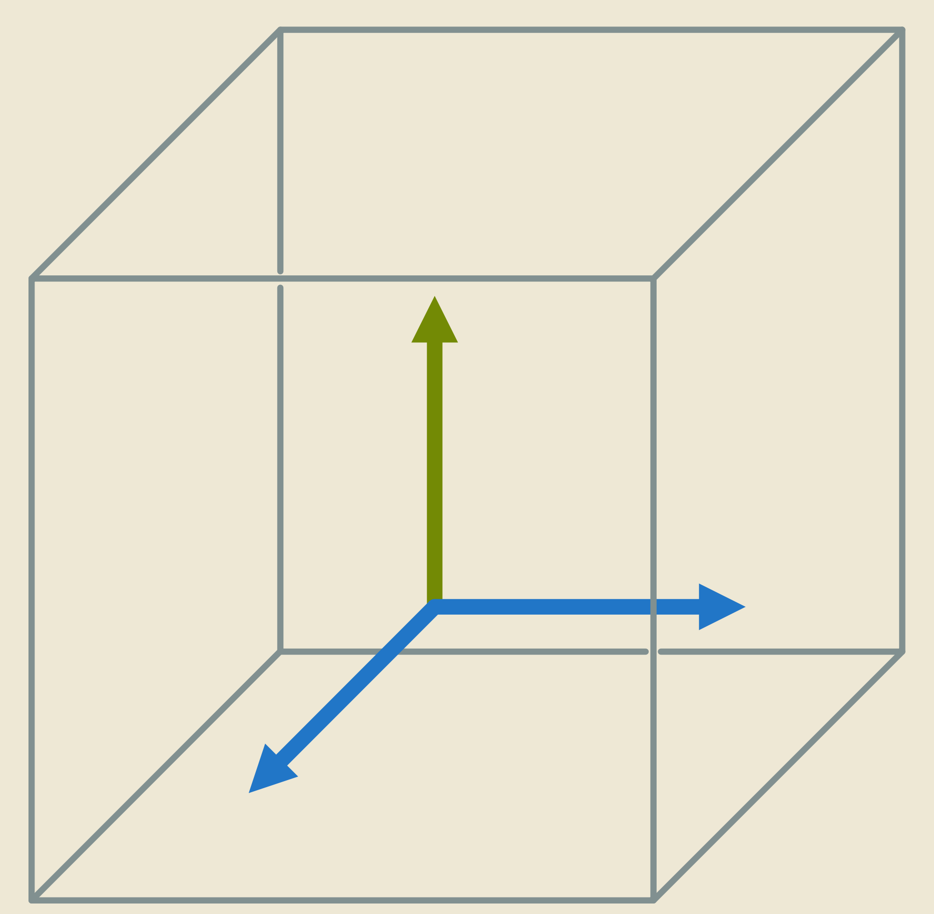
(Boosts ~ ambiguity in Ehresmann connection and horizontal section)

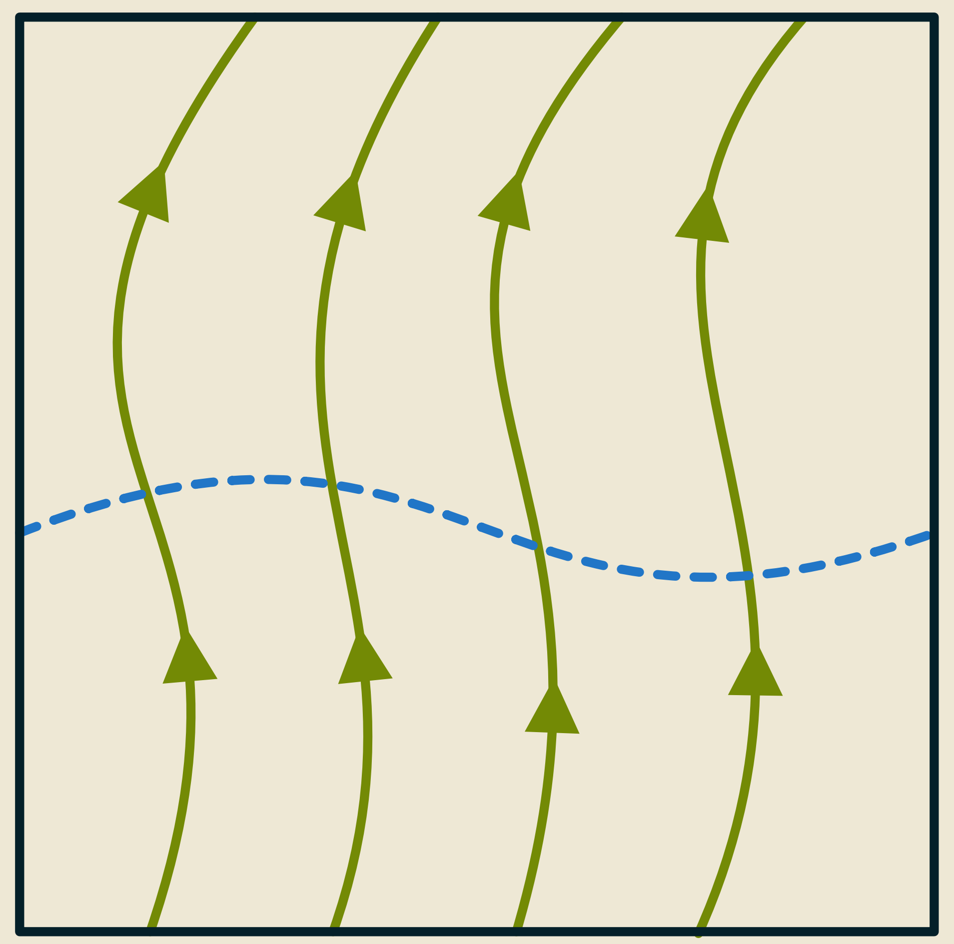
From $c \to 0$ limit get Carroll boost Ward identity

$$0 = T^\mu{}_\nu \left(E^A{}_\mu \Lambda^A{}_B \Theta_B{}^\nu \right) = T^\mu{}_\nu \left(-E^a{}_\mu \frac{1}{c} \Lambda^0{}_a V^\nu + T_\mu c \Lambda^b{}_0 \Theta_b{}^\nu \right) = -\lambda_\mu T^\mu{}_\nu v^\nu + \mathcal{O}(c^2)$$

Since $\lambda_\mu = e^a{}_\mu \lambda_a$ is spatial, this means $T^i{}_0 = 0$, vanishing energy flux

[De Boer, Hartong, Obers, Sybesma, Vandoren]





To boost or not to boost?

Local Carroll boost symmetry implies vanishing energy flux $T^i{}_0 = 0$

inevitable when taking limit of Lorentz-invariant theory

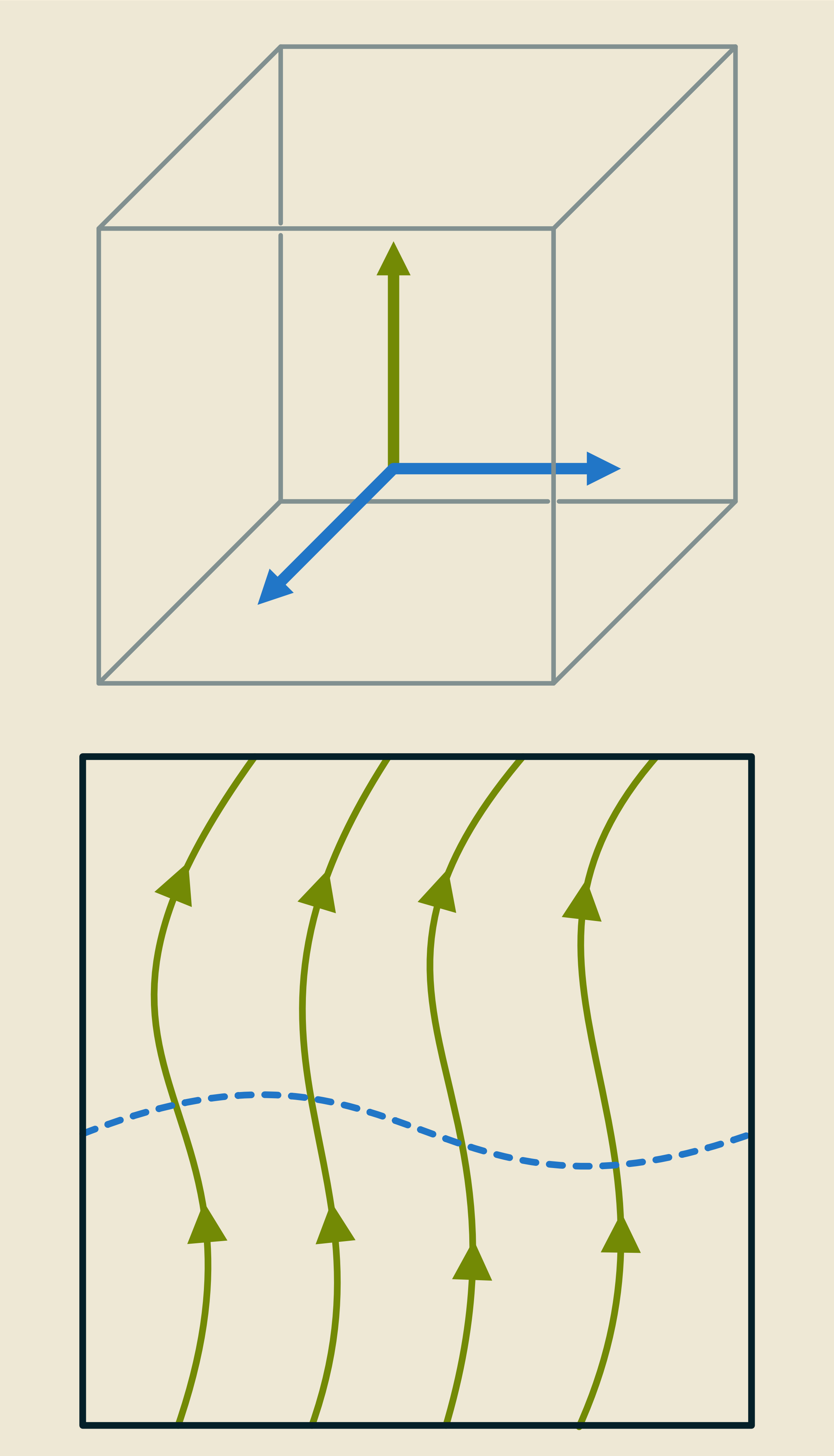
Boosts also constrain $\langle \phi(t,x)\phi(0,0)\rangle = \begin{cases} f(t)\delta(x) \\ g(x) \end{cases}$, 'timelike' and 'spacelike' branches

[Henneaux, Salgado-Rebodello] [De Boer, Hartong, Obers, Sybesma, Vandoren]

Timelike branch reproduces CCFT correlators [Bagchi, Banerjee, Basu, Dutta]

However maybe *Carroll boosts not always desired* in flat space holography?

- known holographic fluids with $T^i{}_0 \neq 0$ [Ciambelli, Marteau, Petkou, Petropoulos, Siampos]
- focus instead on $(v^\mu, h_{\mu\nu})$ fiber structure?
  [Ciambelli, Leigh, Marteau, Siampos] [Petkou, Petropoulos, Rivera Betancour, Siampos] [Freidel, Jai-akson]…
- go to Lorentz-breaking frame before taking flat/Carroll limit in AdS/CFT?
  cf [Campoleoni, Ciambelli, Delfante, Marteau, Petropoulos, Ruzziconi]



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Carroll connection and curvature

For now return to Carroll geometry with boosts, from $c \to 0$ of Lorentzian geometry

Introduce extrinsic curvature $K_{\mu\nu} = -\frac{1}{2}\mathscr{L}_v h_{\mu\nu}$ and acceleration $a_\mu = 2v^\rho \partial_{[\mu}\tau_{\rho]}$

Both boost-invariant and spatial tensors (so $K^{\mu\nu} = h^{\mu\rho}h^{\nu\sigma}K_{\rho\sigma}$ etc)

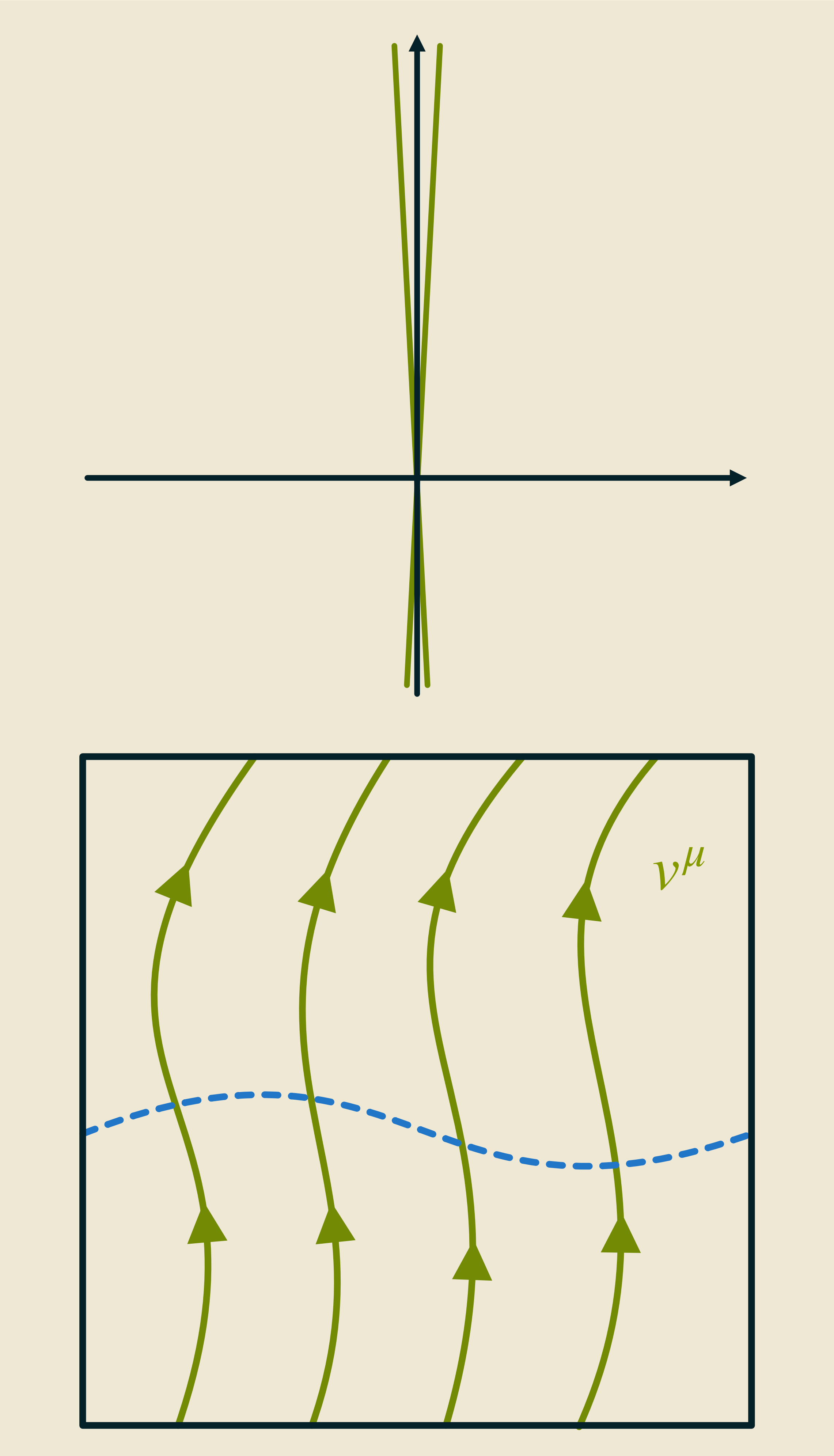
Want connection that satisfies $\tilde{\nabla}_\mu v^\nu = 0$ and $\tilde{\nabla}_\rho h_{\mu\nu} = 0$, choose

$$\tilde{\Gamma}^\rho_{\mu\nu} = -v^\rho \partial_{(\mu}\tau_{\nu)} - v^\rho \tau_{(\mu}\mathscr{L}_v \tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right) - h^{\rho\sigma}\tau_\nu K_{\mu\sigma}$$

Has non-zero torsion $\tilde{T}^\rho{}_{\mu\nu} = 2h^{\rho\sigma}\tau_{[\mu}K_{\nu]\sigma}$ [Bekaert, Morand] [Hansen, Obers, GO, Søgaard]

Define its Riemann curvature in the usual way

$$\tilde{R}_{\mu\nu\sigma}{}^\rho = -\partial_\mu \tilde{\Gamma}^\rho_{\nu\sigma} + \partial_\nu \tilde{\Gamma}^\rho_{\mu\sigma} - \tilde{\Gamma}^\rho_{\mu\lambda}\tilde{\Gamma}^\lambda_{\nu\sigma} + \tilde{\Gamma}^\rho_{\nu\lambda}\tilde{\Gamma}^\lambda_{\mu\sigma}$$



Carroll geometry from Lorentzian

Carroll connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$ can be obtained from Levi-Civita connection,

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{c^2} S_{(-2)}{}^{\rho}{}_{\mu\nu} + \tilde{C}^{\rho}_{\mu\nu} + S_{(0)}{}^{\rho}{}_{\mu\nu} + c^2 S_{(2)}{}^{\rho}{}_{\mu\nu} \,,$$

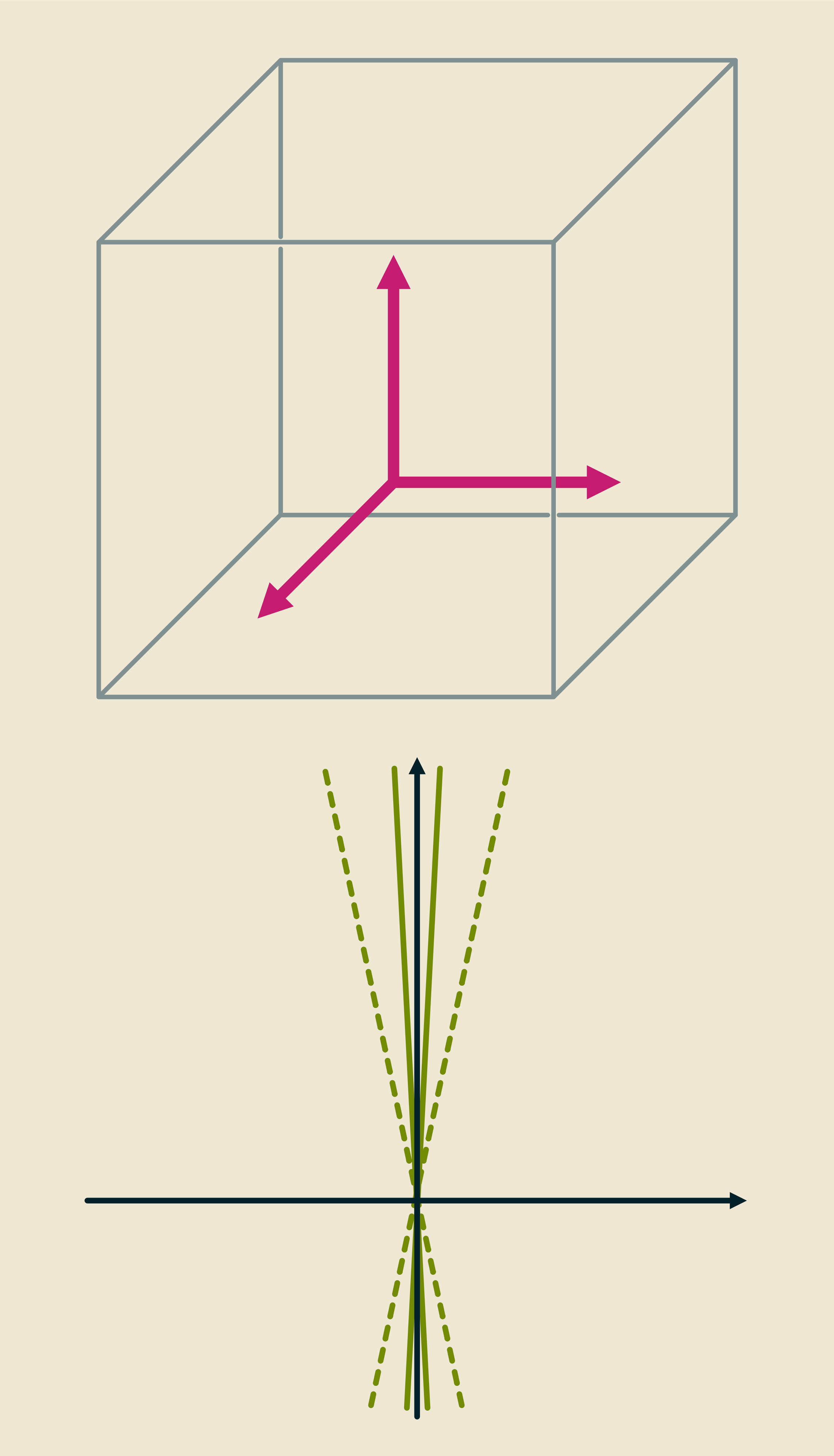
where the $S^{\rho}{}_{\mu\nu}$ are known tensors and $\tilde{C}^{\rho}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu} + \cdots$

Then Levi-Civita Ricci scalar is

$$R = \frac{1}{c^2}\left[\mathscr{K}^{\mu\nu}\mathscr{K}_{\mu\nu} + \mathscr{K}^2 - 2V^{\mu}\partial_{\mu}\mathscr{K}\right] + \Pi^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_{\mu}A^{\mu} + c^2\Pi^{\mu\rho}\Pi^{\nu\sigma}\partial_{[\mu}T_{\nu]}\partial_{[\rho}T_{\sigma]}$$

where $\mathscr{K}_{\mu\nu} = K_{\mu\nu} + \cdots$, $\Pi^{\mu\nu} = h^{\mu\nu} + \cdots$, $\Pi_{\mu\nu} = h_{\mu\nu} + \cdots$ and $A_{\mu} = a_{\mu} + \cdots$

Finally, $\sqrt{-g} = cE = ce + \cdots$ where $E = \det(T_{\mu}, \Pi_{\mu\nu})$ and $e = \det(\tau_{\mu}, h_{\mu\nu})$



Conformal scalar actions: timelike

Rewrite Lorentzian conformal scalar action,

$$S = -\frac{1}{2}\int d^d x \sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{d-2}{4(d-1)}R\phi^2\right)$$

$$= -\frac{c}{2}\int d^d x\, E\left[\left(-\frac{1}{c^2}V^\mu V^\nu + \Pi^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi + \frac{d-2}{4(d-1)}\left(\frac{1}{c^2}\left[\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} + \mathcal{K}^2 - 2V^\mu\partial_\mu\mathcal{K}\right] + \Pi^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_\mu A^\mu + c^2\Pi^{\mu\rho}\Pi^{\nu\sigma}\partial_{[\mu}T_{\nu]}\partial_{[\rho}T_{\sigma]}\right)\phi^2\right]$$

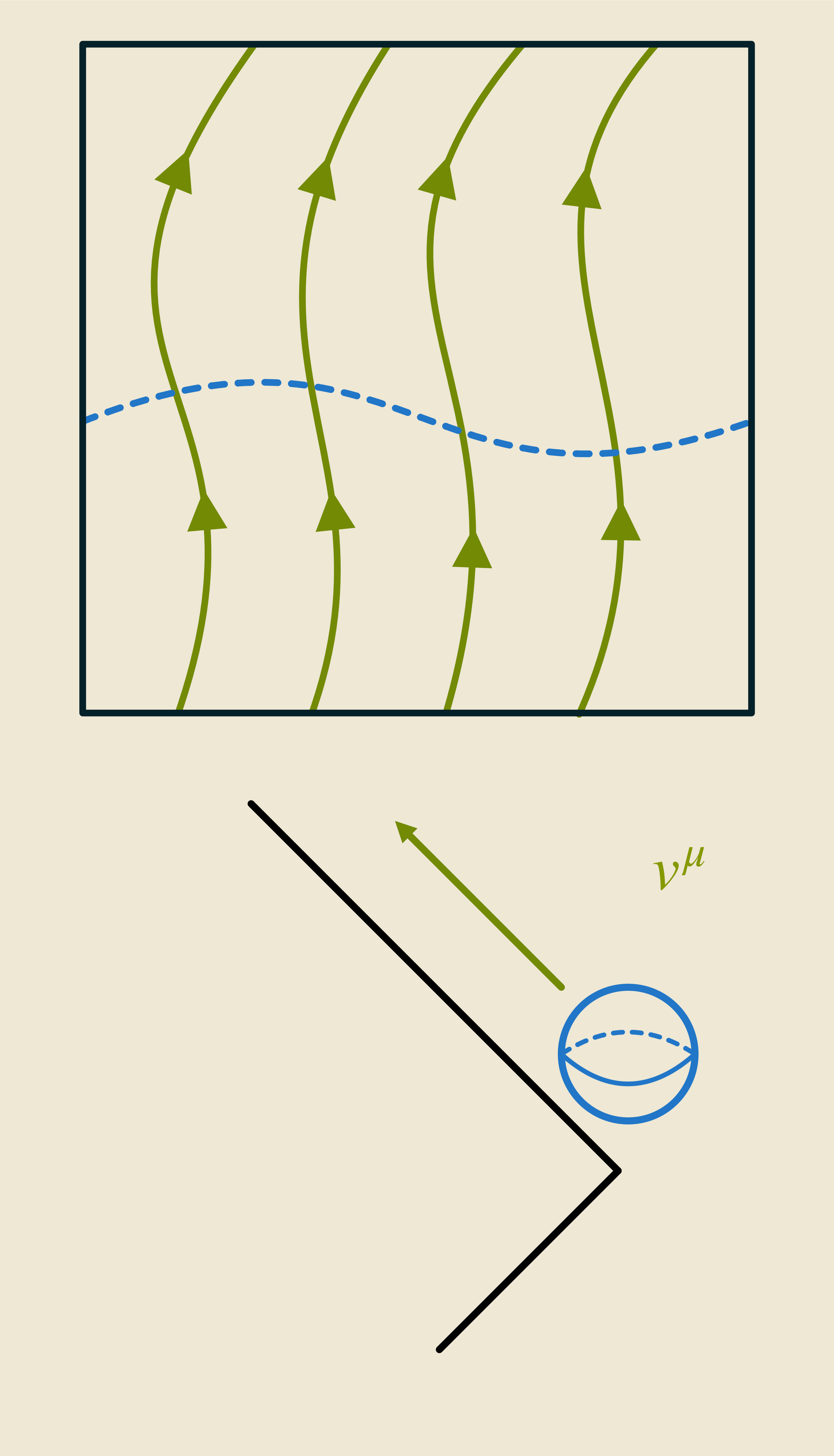
In Carroll limit $c \to 0$, leading-order terms give [Baiguera, GO, Sybesma, Søgaard]

$$S_t = -\frac{1}{2}\int d^d x\, e\left[-(v^\mu\partial_\mu\phi)^2 + \frac{(d-2)}{4(d-1)}\left(K^{\mu\nu}K_{\mu\nu} + K^2 - 2v^\mu\partial_\mu K\right)\phi^2\right]$$

This is timelike conformal Carroll scalar, flat space propagator $\sim t\,\delta(x)$

Carroll boost-invariant and Weyl-invariant, so $T^i{}_0 = 0$ and $T^\mu{}_\mu = 0$

Also considered from no-boost approach in [Gupta, Suryanarayana] [Rivera-Betancour, Vilatte]



Conformal scalar actions: spacelike

Rewrite Lorentzian conformal scalar action,

$$S = -\frac{1}{2}\int d^d x \sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{d-2}{4(d-1)}R\phi^2\right)$$

$$= -\frac{c}{2}\int d^d x\, E\left[\left(-\frac{1}{c^2}V^\mu V^\nu + \Pi^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi\right.$$

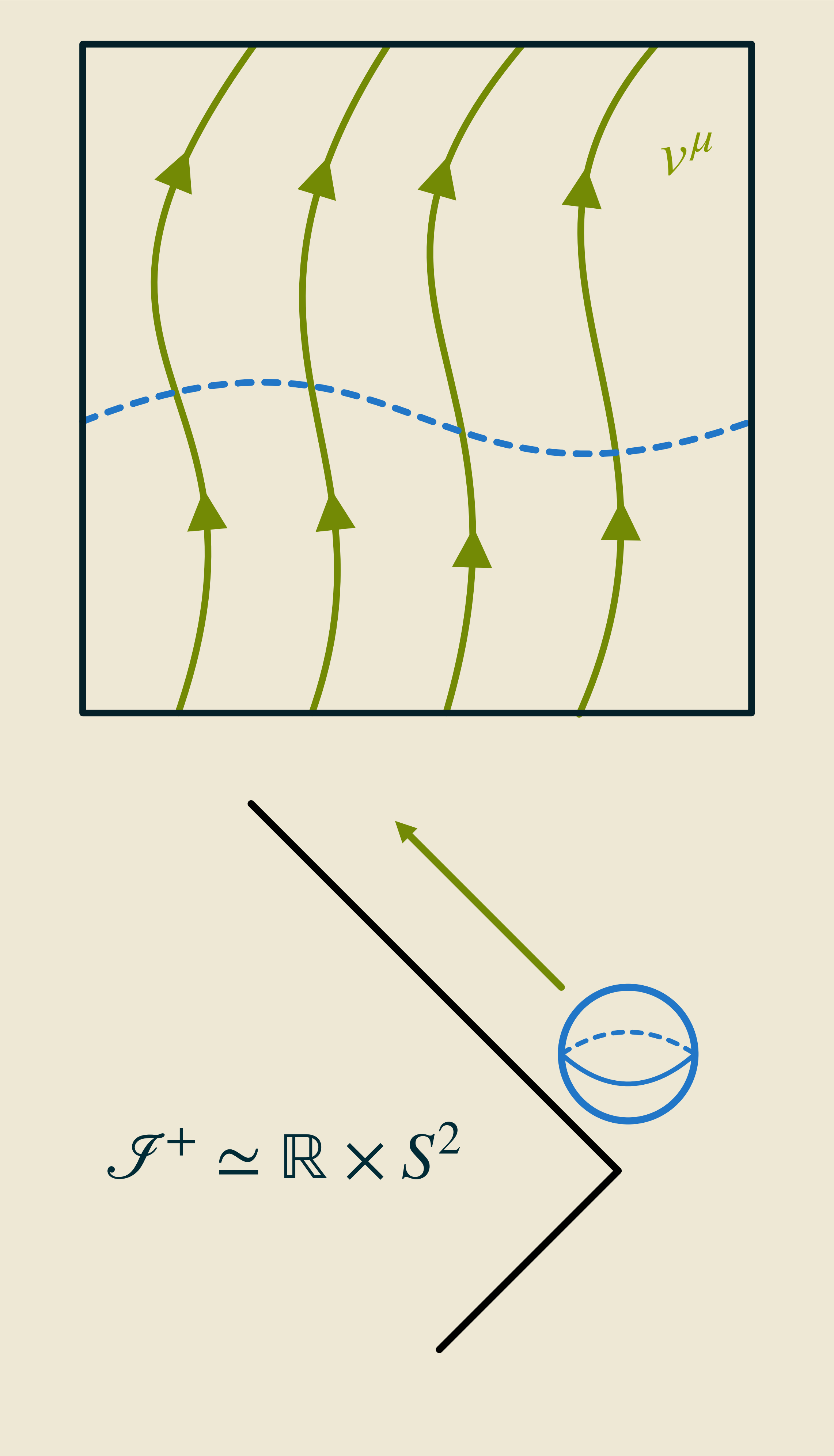
$$\left. + \frac{d-2}{4(d-1)}\left(\frac{1}{c^2}\left[\mathscr{K}^{\mu\nu}\mathscr{K}_{\mu\nu} + \mathscr{K}^2 - 2V^\mu\partial_\mu\mathscr{K}\right] + \Pi^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_\mu A^\mu + c^2\Pi^{\mu\rho}\Pi^{\nu\sigma}\partial_{[\mu}T_{\nu]}\partial_{[\rho}T_{\sigma]}\right)\phi^2\right]$$

Can take alternative Carroll limit $c \to 0$ using Lagrange multipliers,

$$S_s = -\frac{1}{2}\int d^d x\, e\left[h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{(d-2)}{4(d-1)}\left(h^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_\mu a^\mu\right) + \chi\left(v^\mu\partial_\mu\phi + \frac{(d-2)}{4(d-1)}K\right) + \chi^{\mu\nu}\breve{K}_{\mu\nu}\phi\right]$$

This is spacelike conformal Carroll scalar. [Baiguera, GO, Sybesma, Søgaard]

- also invariant under boosts and Weyl transformations
- extrinsic curvature must be pure trace $K_{\mu\nu} = \dfrac{h_{\mu\nu}}{d-1}K$
- time-dependence $v^\mu\partial_\mu\phi$ is fixed, so only spacelike dynamics



Conformal scalar actions: spacelike

Spacelike conformal Carroll scalar from next-to-leading-order terms,

$$S_s = -\frac{1}{2}\int d^d x\, e\left[h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{(d-2)}{4(d-1)}\left(h^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_\mu a^\mu\right)\phi^2 + \chi\left(v^\mu\partial_\mu\phi + \frac{(d-2)}{4(d-1)}K\right) + \chi^{\mu\nu}\check{K}_{\mu\nu}\phi\right]$$

- invariant under local Carroll boosts and Weyl transformations
- energy-momentum tensor satisfies ${T^i}_0 = 0$ and ${T^\mu}_\mu = 0$

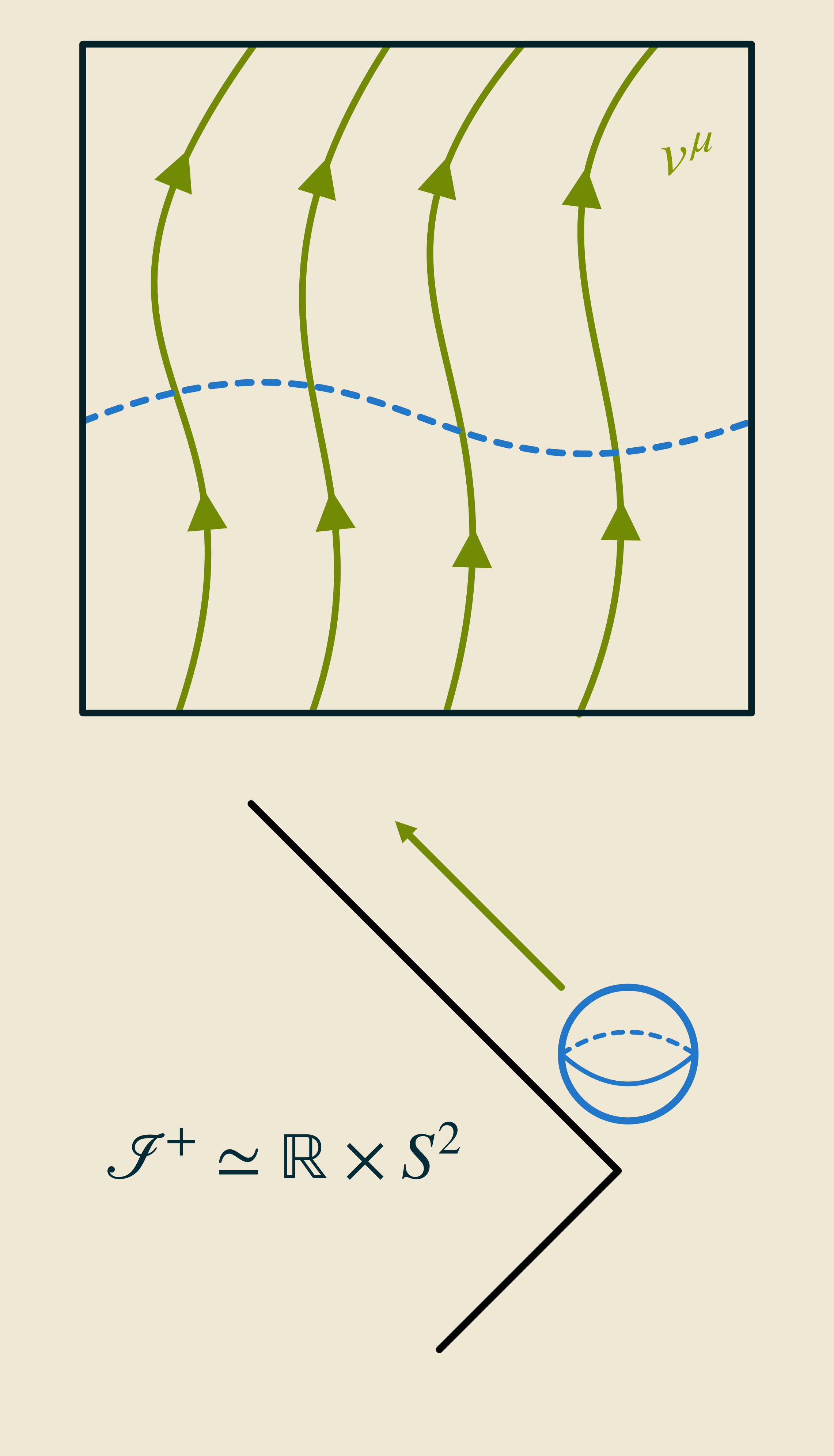
Flat space propagator $\sim \log(x)^2$ of spacelike Euclidean free boson

Remarkably, can dimensionally reduce the action explicitly using constraints,

$$S_s = -\frac{1}{2}\int d^{d-1}x\sqrt{h}\left(h^{ij}\partial_i\hat{\phi}\partial_j\hat{\phi} + \frac{(d-3)}{4(d-2)}\hat{R}\hat{\phi}^2 + \frac{1}{4(d-1)(d-2)}A^{-2}\hat{R}_{A^{-2}h_{ij}}\right)$$

where $\hat{\phi} = A^{1/2}\phi$ and the background field $A = \int_v \tau$ encodes former 'Carroll time'

but otherwise this is $(d-1)$-dimensional Euclidean conformal scalar!



Conformal scalar actions: spacelike

Dimensional reduction to Euclidean theory

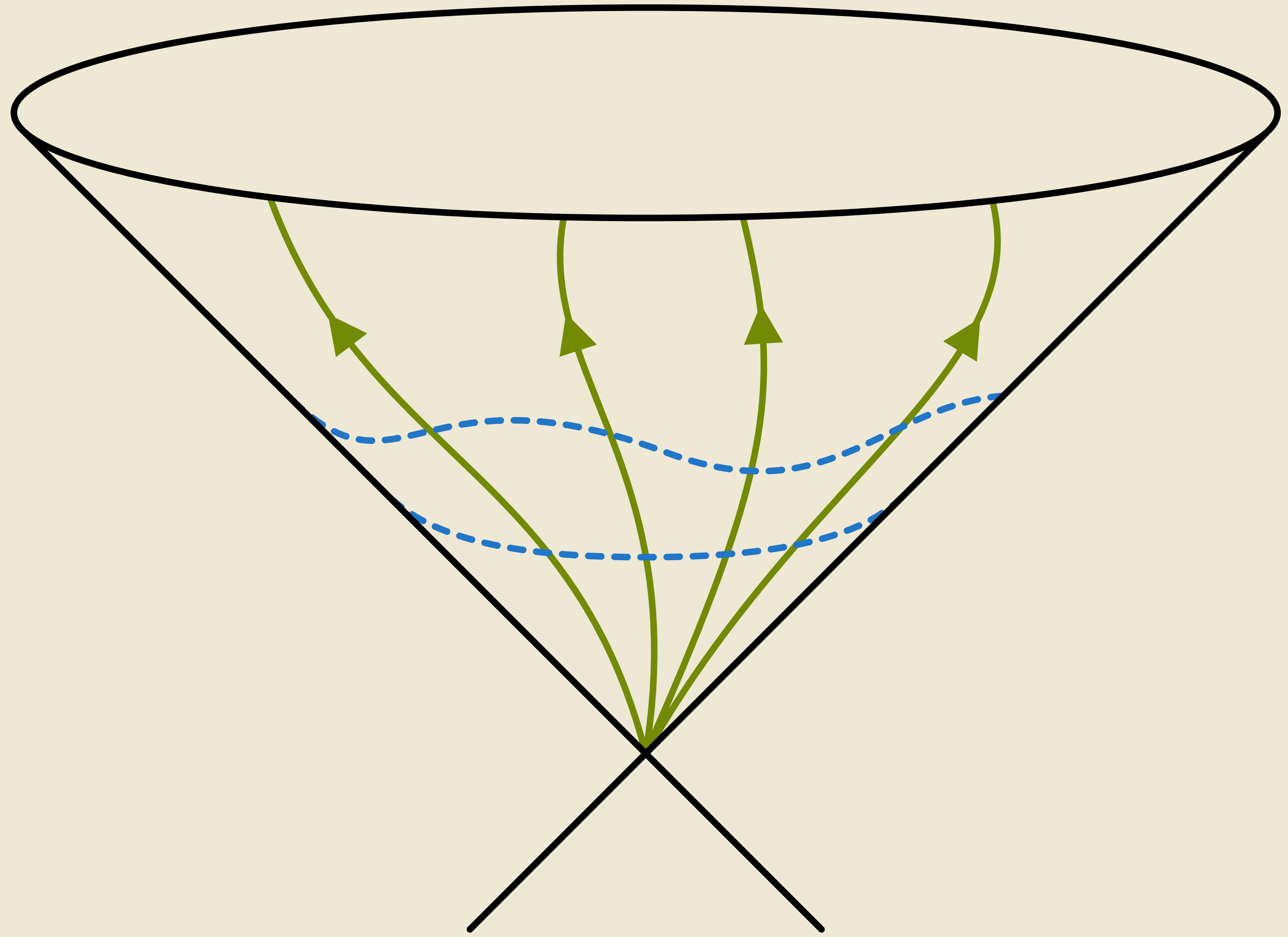
$$S_s = -\frac{1}{2}\int d^d x\, e \left[h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{(d-2)}{4(d-1)}\left(h^{\mu\nu}\tilde{R}_{\mu\nu} - \tilde{\nabla}_\mu a^\mu\right)\phi^2 + \chi\left(v^\mu\partial_\mu\phi + \frac{(d-2)}{4(d-1)}K\right) + \chi^{\mu\nu}\breve{K}_{\mu\nu}\phi\right]$$
$$= -\frac{1}{2}\int d^{d-1}x\sqrt{h}\,\left(h^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} + \frac{(d-3)}{4(d-2)}\hat{R}\hat{\phi}^2 + \frac{1}{4(d-1)(d-2)}A^{-2}\hat{R}_{A^{-2}h_{ij}}\right)$$

Reminiscent of embedding space formalism!

Get $(d-1)$-dim conformal $SO(d,1)$ representations from $(d+1)$-dim Lorentz representations in $\mathbb{R}^{1,d}$

Restriction to light cone
- $\Longrightarrow$ Carrollian spacelike theory
- $\Longrightarrow$ Euclidean theory

Similar procedure for other spacelike Carroll theories?



Conformal Carroll anomalies

Can geometrically classify all possible Weyl anomalies

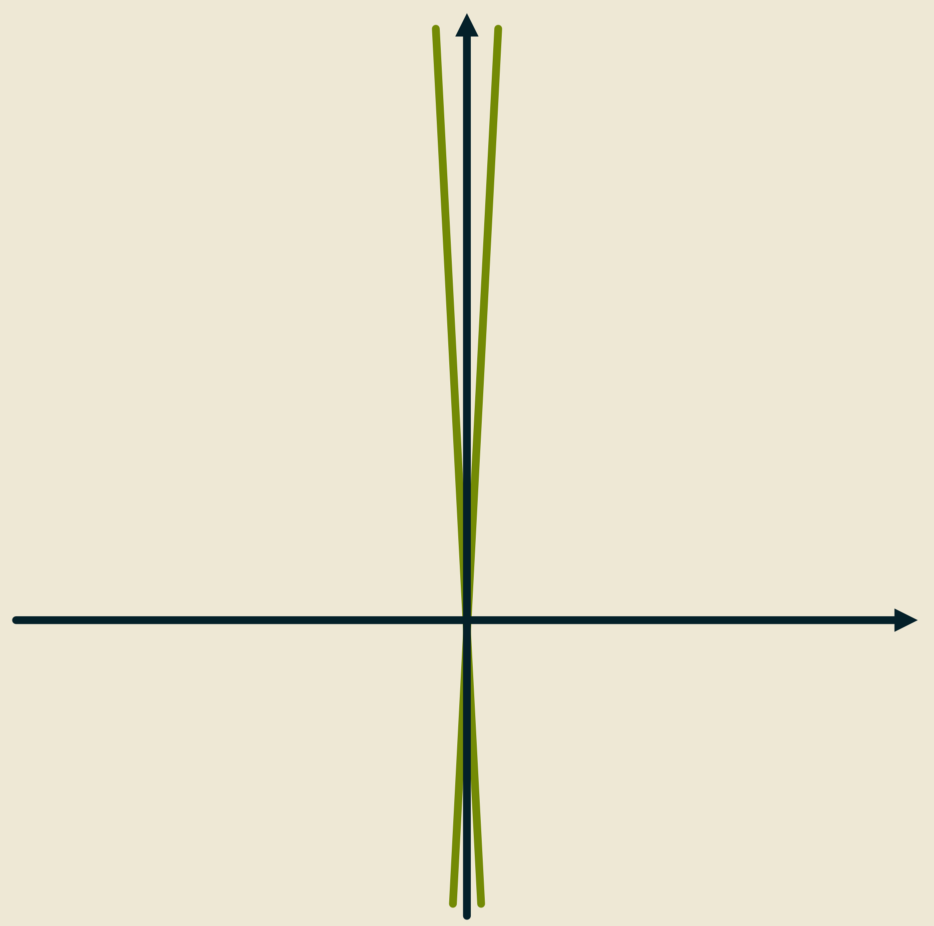
In Lorentzian case, find $\langle T^{\mu}{}_{\mu}\rangle = \begin{cases} -\frac{c}{24\pi}R & d=2, \\ aE_4 - cW^2 & d=4, \\ & \vdots \end{cases}$

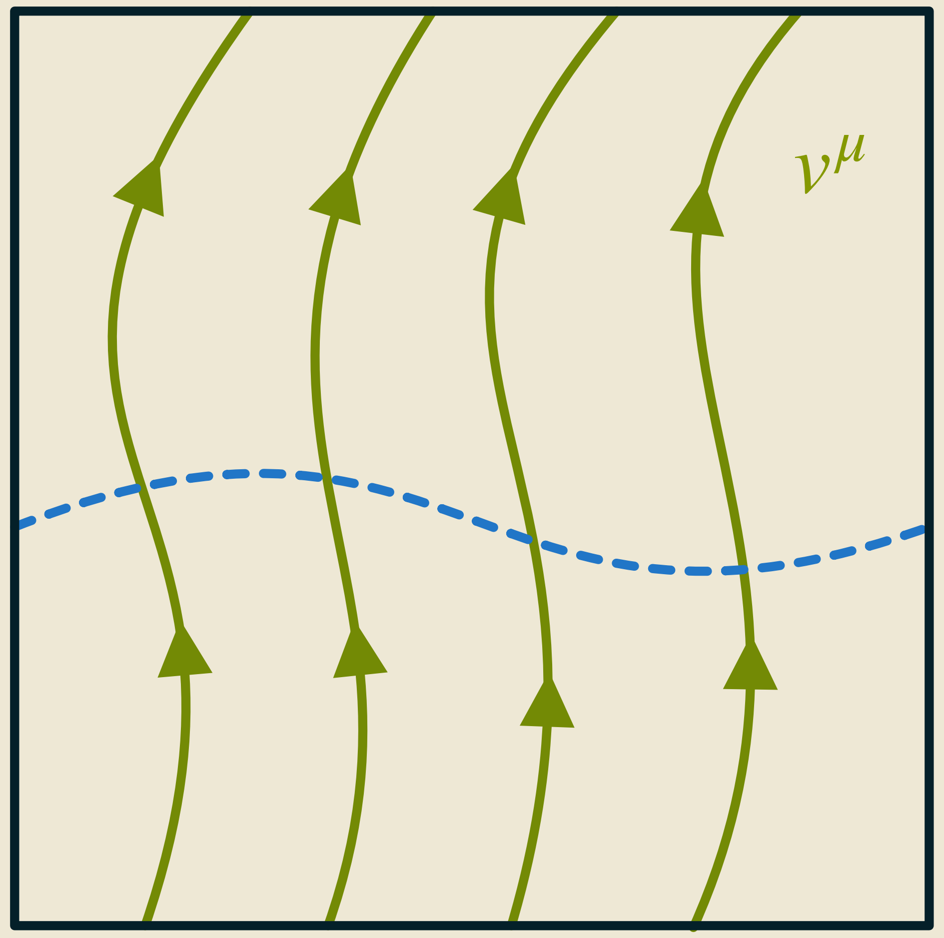
In Carrollian case, have different connection and curvature, so other invariants

Timelike anomalies: using only v^{μ} , $h_{\mu\nu}$, $\mathscr{L}_v$ and $K_{\mu\nu}$ get

$$\langle T^{\mu}{}_{\mu}\rangle = \begin{cases} \varnothing & d=2 \\ \varnothing & d=3 \\ b_1\left(-7\mathrm{Tr}(K^4) + \frac{1}{3}K\mathrm{Tr}(K^3) + \mathrm{Tr}(K^2)(\mathscr{L}_v K) + (\mathscr{L}_v K)^2\right) + b_2(\cdots) & d=4, \\ & \vdots \end{cases}$$

[Baiguera, GO, Sybesma, Søgaard] [Arav, Chapman, Oz]





No 3d Carroll anomalies $\implies$ no 2d CCFT anomalies $\sim$ celestial $T_{\mu\nu}$ not renormalized?

Summary and outlook

Constructed timelike and spacelike conformal Carroll scalar actions

Allow explicit computations using only basic QFT techniques

Ongoing and future work:

- study sources and breaking of boosts ~ supertranslations
- complete general anomaly classification
- fermions?
- direct computation of scalar anomalies?

Build up conformal Carroll $\Longleftrightarrow$ CCFT dictionary

Top-down flat holography from $c \to 0$ limit of AdS/CFT?