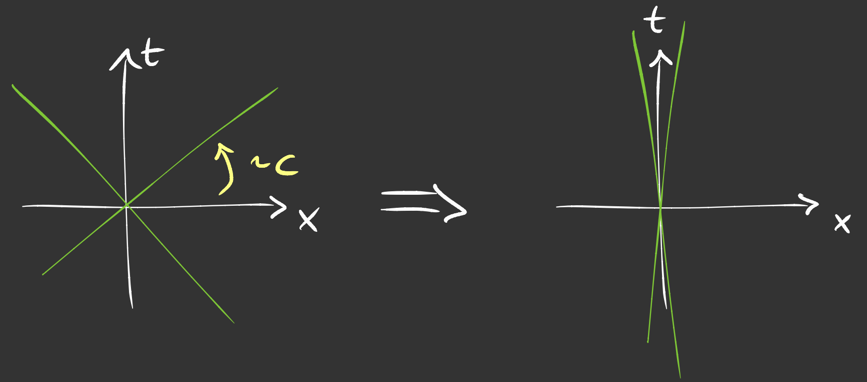
Carroll Expansion of General Relativity



Vienna Carroll Meeting

Gerben Oling, Nordita

16-02-2022

based on 2112.12684 with

Dennis Hansen, Niels Obers,

Benjamin Søgaard

Outline

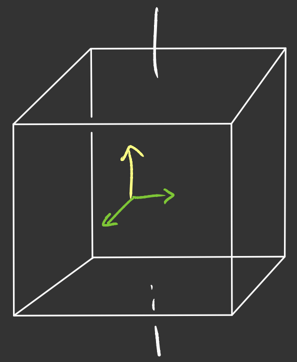
$$S_{EH} = c^2 S_{LO} + c^4 S_{NLO} + \dots$$

- review pre-ultra-local parametrization
- leading-order action = electric theory
- next-to-leading action ⊃ magnetic theory
- solutions of constraints and evolution equations

Prepare:

write GR in suitable variables

[Hansen, Hartong, Obers, GO, Søgaard]



$$g_{\mu\nu} = \eta_{AB} E^A_\mu E^B_\nu = -c^2 T_\mu T_\nu + \Pi_{\mu\nu} \qquad \Pi_{\mu\nu} = \delta_{ab} E^a_\mu E^b_\nu$$

"time" "space"

$$g^{\mu\nu} = \eta^{AB} E^\mu_A E^\nu_B = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu} \qquad \Pi^{\mu\nu} = \delta^{ab} E^\mu_a E^\nu_b$$

$$V^\mu T_\mu = -1, \quad V^\mu \Pi_{\mu\nu} = 0, \quad T_\mu \Pi^{\mu\nu} = 0, \quad \delta^\mu_\nu = -V^\mu T_\nu + \Pi^{\mu\rho}\Pi_{\rho\nu}$$

Expanding: $V^\mu = v^\mu + c^2 M^\mu + \dots \qquad T_\mu = \tau_\mu + \dots$

$$\Pi^{\mu\nu} = h^{\mu\nu} + c^2 \Phi^{\mu\nu} + \dots \qquad \Pi_{\mu\nu} = h_{\mu\nu} + \dots$$

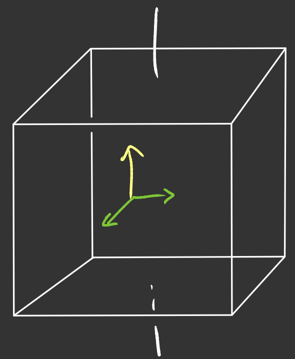
Local Lorentz $\Longrightarrow$ Carroll at LO, plus corrections at NLO

$$\delta v^\mu = 0 \qquad \delta \tau_\mu = \lambda_\mu$$

$$\delta h^{\mu\nu} = v^\mu \lambda^\nu + v^\nu \lambda^\mu \qquad \delta h_{\mu\nu} = 0$$

where $v^\mu \lambda_\mu = 0$ and $\lambda^\mu = h^{\mu\nu}\lambda_\nu$

Prepare: write GR in suitable variables



$$V^\mu = v^\mu + c^2 M^\mu + \dots \qquad T_\mu = \tau_\mu + \dots$$

$$\Pi^{\mu\nu} = h^{\mu\nu} + c^2 \Phi^{\mu\nu} + \dots \qquad \Pi_{\mu\nu} = h_{\mu\nu} + \dots$$

Want adapted $\tilde{\Gamma}^\rho_{\mu\nu}$ satisfying $\tilde{\nabla}_\mu v^\nu = 0$ and $\tilde{\nabla}_\rho h_{\mu\nu} = 0$

For this, use adapted $\tilde{C}^\rho_{\mu\nu}$ satisfying $\overset{(\tilde{C})}{\nabla}_\mu V^\nu = 0$ and $\overset{(\tilde{C})}{\nabla}_\rho \Pi_{\mu\nu} = 0$

— already before expansion!

'minimal' torsion solution

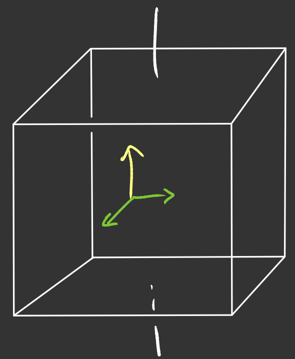
$$\tilde{C}^\rho_{\mu\nu} = -V^\rho \partial_{(\mu} T_{\nu)} - V^\rho T_{(\mu} \mathcal{L}_V T_{\nu)} + \frac{1}{2} \Pi^{\rho\lambda} \left[\partial_\mu \Pi_{\nu\lambda} + \partial_\nu \Pi_{\lambda\mu} - \partial_\lambda \Pi_{\mu\nu} \right] - \Pi^{\rho\lambda} T_\nu K_{\mu\lambda}$$

see also [Bekaert, Morand] [Hartong]

extrinsic curvature $K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_V \Pi_{\mu\nu}$, satisfies $V^\mu K_{\mu\nu} = 0$

determines torsion $2\tilde{C}^\rho_{[\mu\nu]} = \Pi^{\rho\lambda} T_{[\mu} K_{\nu]\lambda}$

Prepare: write GR in suitable variables



$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu} , \quad g^{\mu\nu} = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}$$

$\tilde{C}^{\rho}_{\mu\nu}$ satisfying $\overset{(\tilde{C})}{\nabla}_\mu V^\nu = 0$ and $\overset{(\tilde{C})}{\nabla}_\rho \Pi_{\mu\nu} = 0$

"pre-ultra-local" parametrization of Einstein-Hilbert:

$$S_{EH} = \frac{c^3}{16\pi G} \int_M R \sqrt{-g}\, d^{d+1}x$$

$$= \frac{c^2}{16\pi G} \int_M \left[\left(K^{\mu\nu} K_{\mu\nu} - K^2\right) + c^2 \Pi^{\mu\nu} \overset{(\tilde{C})}{R}_{\mu\nu} + \frac{c^4}{4} \Pi^{\mu\nu} \Pi^{\rho\sigma} (dT)_{\mu\rho} (dT)_{\nu\sigma} \right] E\, d^{d+1}x$$

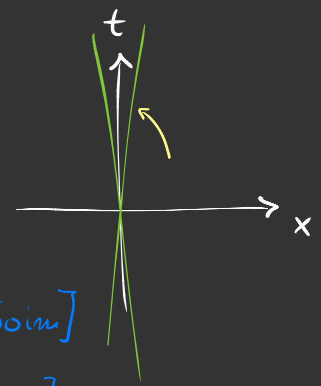
up to boundary terms

[Hansen, Obers, GO, Søgaard]

From here can expand in (even) powers of c^2 $\Longrightarrow$ $S_{EH} = c^2 S_{LO} + c^4 S_{NLO} + \dots$

LO:

$$S_{EH} = c^2 S_{LO} + \dots$$



Gives $$S_{LO} = \frac{1}{16\pi G}\int_M \left(K^{\mu\nu} K_{\mu\nu} - K^2 \right) e \, d^{d+1}x ,$$ cf [Isham] [Teitelboim] [Henneaux] [Hartong]

where $K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_v h_{\mu\nu}$ is extrinsic curvature

Reproduces electric Carroll limit of general relativity! [Hansen, Obers, GO, Søgaard]

Equations of motion

$$\delta v^\mu \Rightarrow 0 = -\frac{1}{2}\tau_\mu \left(K^{\rho\sigma} K_{\rho\sigma} - K^2 \right) + h^{\gamma\lambda} \tilde{\nabla}_\lambda \left(K_{\mu\gamma} - K h_{\mu\gamma} \right)$$

$$\delta h^{\mu\nu} \Rightarrow 0 = -\frac{1}{2} h_{\mu\nu} \left(K^{\rho\sigma} K_{\rho\sigma} - K^2 \right) + K \left(K_{\mu\nu} - K h_{\mu\nu} \right) - v^\rho \tilde{\nabla}_\rho \left(K_{\mu\nu} - K h_{\mu\nu} \right)$$

Can split into

$$K^{\mu\nu} K_{\mu\nu} - K^2 = 0$$
$$h^{\rho\sigma} \tilde{\nabla}_\rho \left(K_{\sigma\mu} - K h_{\sigma\mu} \right) = 0$$

| constraint equations

$$\mathcal{L}_v K_{\mu\nu} = -2 K_\mu{}^\rho K_{\rho\nu} + K K_{\mu\nu}$$

| evolution equations

LO:

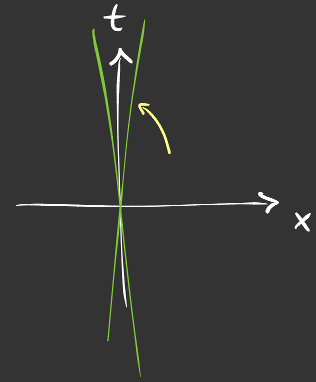
$$K^{\mu\nu} K_{\mu\nu} - K^2 = 0$$

$$h^{\rho\sigma} \tilde{\nabla}_\rho \left(K_{\sigma\mu} - K h_{\sigma\mu} \right) = 0$$

constraint equations

$$\mathcal{L}_v K_{\mu\nu} = -2 K_\mu{}^\rho K_{\rho\nu} + K K_{\mu\nu}$$

evolution equations



Evolution of initial data $(h^0_{\mu\nu}, K^0_{\mu\nu})$

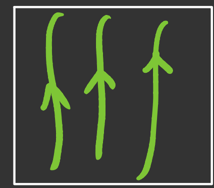
depends only on v^μ derivatives $\rightarrow$ ultra-local!

In adapted coordinates $x^\mu = (t, x^i)$ take $v = e^{-\frac{1}{2} \dot{h}^{ij} h_{ij}} \partial_t$

then can integrate $\boxed{h_{ij}(t) = h^0_{ik}\, e^{-2t\, h_0^{kl} K^0_{lj}}}$

for arbitrary initial data [Hansen, Obers, GO, Søgaard]

see also [Dautcourt], [Niedermaier]



Much simpler than evolution in full GR!

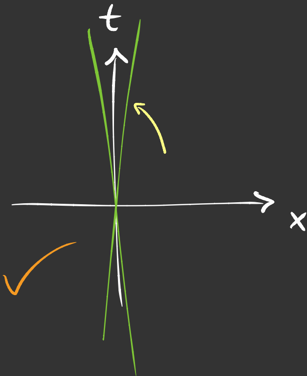
LO:

$$K^{\mu\nu}K_{\mu\nu} - K^2 = 0$$
$$h^{\rho\sigma}\tilde{\nabla}_\rho\left(K_{\sigma\mu} - K h_{\sigma\mu}\right) = 0$$

constraint equations

$$\mathcal{L}_v K_{\mu\nu} = -2K_\mu{}^\rho K_{\rho\nu} + K K_{\mu\nu}$$

evolution equations ✓



Construct initial data using 3+1 methods ~ Bowen – York type solutions

$$h^0_{ij} = \psi^4\delta_{ij}$$

$$K^0_{ij} = \psi^{-2}\bar{L}X_{ij} + \frac{1}{3}K_0\psi^4\delta_{ij}$$

where $\psi = \left[\frac{3}{2K_0^2}\bar{L}X_{ij}\bar{L}X^{ij}\right]^{1/12}$

$$\bar{L}X^{ij} = \frac{3}{2r^2}\left[x^iP^j + x^jP^i - \left(\delta^{ij} - \frac{x^ix^j}{r^2}\right)P_kx^k\right] + \frac{3}{r^5}\left[\varepsilon^{ik}{}_\ell J_k x^\ell x^j - \varepsilon^{jk}{}_\ell J_k x^\ell x^i\right]$$

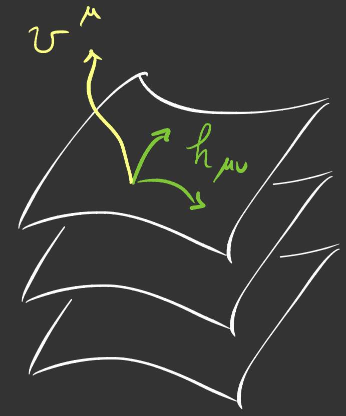
Have physical boundary charges $(\vec{P}, \vec{J})$ [Hansen, Obers, GO, Søgaard] [Perez]

$$\Theta^\mu = \frac{e}{8\pi G}\left[\left(K h^\mu_\nu - K^\mu{}_\nu\right)\delta v^\nu - \frac{1}{2}\left(K h_{\sigma\rho} - K_{\sigma\rho}\right)v^\mu\delta h^{\sigma\rho}\right]$$

$$Q^{[\mu\nu]} = \frac{e}{4\pi G}\left(v^{[\mu}K^{\nu]}{}_\sigma\xi^\sigma - v^{[\mu}\xi^{\nu]}K\right)$$

$\longrightarrow$ no 'mass' charge for $\xi^\mu \sim v^\mu$ at LO !

Intermezzo: Carroll data v^μ and $h_{\mu\nu}$ does not naturally define spatial hypersurface foliation



Can use τ_μ and $h^{\mu\nu}$ in fixed boost frame to define

$$h^\mu{}_\nu = h^{\mu\rho} h_\nu \quad \text{and} \quad -v^\mu \tau_\nu \,, \quad \text{space/time projectors}$$

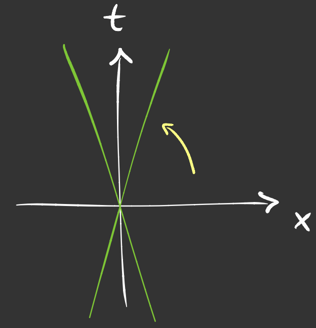
Hypersurface derivative $\hat{\nabla}_\mu X^\nu{}_\rho = h^\alpha_\mu h^\nu_\beta h^\gamma_\rho \tilde{\nabla}_\alpha X^\beta{}_\gamma$

is Levi-Civita, curvature $\hat{R}_{\mu\nu}$ from Gauss-type relations

Can do hypersurface computations in this frame

to solve boost-invariant equations $\Rightarrow$ valid solutions!

NLO: $S_{EH} = c^2 S_{LO} + c^4 S_{NLO} + \dots$



$$S_{NLO} = \frac{1}{16\pi G}\int_M \left[h^{\mu\nu}\tilde{R}_{\mu\nu} + 2\,\overset{(2)}{G}{}^{\nu}_{\mu} M^{\mu} + \frac{1}{2}\,\overset{(2)}{G}{}^{k}_{\mu\nu}\,\Phi^{\mu\nu}\right] e\, d^{d+1}x$$

LO EOM

Full NLO equations of motion are complicated ...

$\longrightarrow$ simplify by setting $M^{\mu} = 0$ and $\Phi^{\mu\nu} = 0$

Only consistent subsector of LO theory if LO EOM also hold

$\longrightarrow$ also set $K_{\mu\nu} = 0$ so that $\overset{(2)}{G}{}^{\nu}_{\mu} = 0$ and $\overset{(2)}{G}{}^{k}_{\mu\nu} = 0$

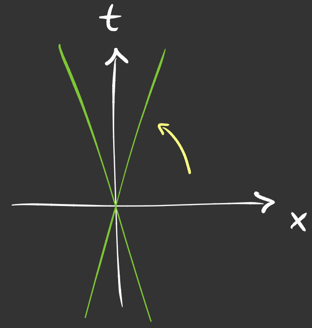
Restricted NLO theory

$$\boxed{S_{\overline{NLO}} = \frac{1}{16\pi G}\int_M \left[h^{\mu\nu}\tilde{R}_{\mu\nu} + \phi^{\mu\nu}K_{\mu\nu}\right] e\, d^{d+1}x}$$

Claim: this is magnetic theory [Henneaux, Salgado] [Perez] !

NLO:

$$S_{\overline{NLO}} = \frac{1}{16\pi G}\int_M \left[h^{\mu\nu}\tilde{R}_{\mu\nu} + \phi^{\mu\nu}K_{\mu\nu}\right] e\, d^{d+1}x$$



Can also derive this $\overline{NLO}$ action from limit, as in field theory

[de Boer, Hartong, Obers, Sybesma]
[Henneaux, Salgado]

$$S_{EH} = \frac{c^2}{16\pi G}\int_M \left[\left(K^{\mu\nu}K_{\mu\nu} - K^2\right) + c^2\, \Pi^{\mu\nu} \overset{(\tilde{c})}{R}_{\mu\nu} + \frac{c^4}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}(d\tau)_{\mu\rho}(d\tau)_{\nu\sigma}\right] E\, d^{d+1}x$$

$$= \frac{c^4}{16\pi G}\int_M \left[-\frac{c^2}{4} G^{\mu\nu\rho\sigma}\chi_{\mu\nu}\chi_{\rho\sigma} + G^{\mu\nu\rho\sigma}\chi_{\mu\nu}K_{\rho\sigma} + \Pi^{\mu\nu}\overset{(\tilde{c})}{R}_{\mu\nu} + \frac{c^2}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}(d\tau)_{\mu\rho}(d\tau)_{\nu\sigma}\right] E\, d^{d+1}x$$

after integrating out $\chi_{\mu\nu}$

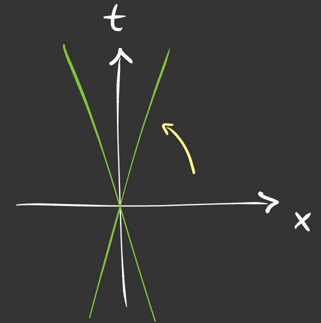
$G^{\mu\nu\rho\sigma} = \frac{1}{2}\left(\Pi^{\mu\rho}\Pi^{\nu\sigma} + \Pi^{\mu\sigma}\Pi^{\nu\rho} - 2\Pi^{\mu\nu}\Pi^{\rho\sigma}\right)$ is "DeWitt" metric

Limit $c \to 0$ then gives $S_{\overline{NLO}}$ with $\phi^{\mu\nu} = G^{\mu\nu\rho\sigma}\chi_{\rho\sigma}$!

[Hansen, Obers, GO, Søgaard]

NLO:

$$S_{NLO} = \frac{1}{16\pi G}\int_M \left[h^{\mu\nu}\tilde{R}_{\mu\nu} + \phi^{\mu\nu}K_{\mu\nu}\right] e\, d^{d+1}x$$



Equations of motion can be written as

$$0 = h^{\mu\nu}\hat{R}_{\mu\nu}$$

$$0 = \hat{\nabla}_\nu \phi^{\nu}{}_{\mu}$$

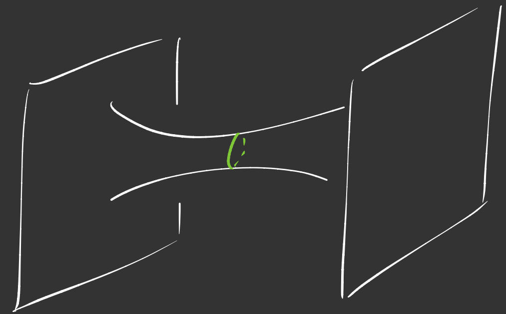
magnetic constraint equations

+

$$\frac{1}{2} h_\nu \phi^{\mu\nu} = \hat{R}_{\mu\nu} - \hat{\nabla}_{(\mu} a_{\nu)} - a_\mu a_\nu + h_{\mu\nu} h^{\rho\sigma}\left(\hat{\nabla}_\rho a_\sigma + a_\rho a_\sigma\right)$$

evolution equation

Due to spatial Ricci tensor $\hat{R}_{\mu\nu}$ now can have massive solutions

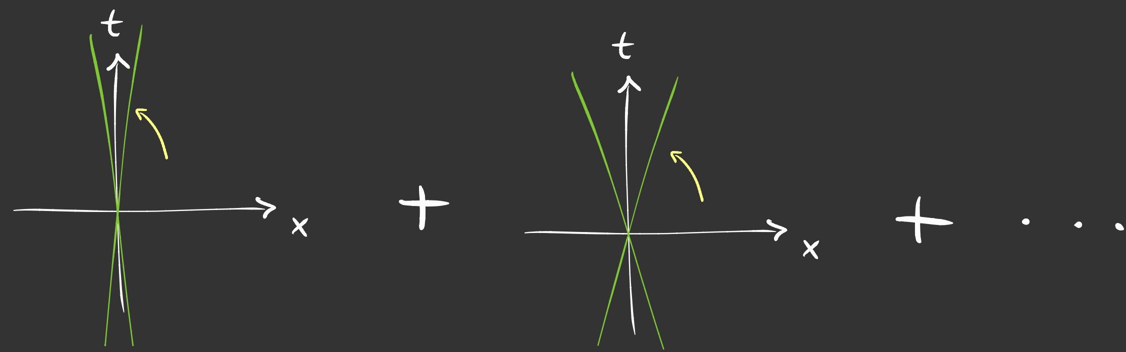


$$v^\mu \partial_\mu = \frac{M+2\rho}{M-2\rho}\partial_t\,, \qquad h_{\mu\nu} dx^\mu dx^\nu = \left(1+\frac{M}{2\rho}\right)^4 \delta_{ij}\, dx^i dx^j$$

~ Schwarzschild isotropic coordinates

[Hansen, Obers, GO, Søgaard] [Perez]

Outlook



Ultra-local expansion of GR

LO = electric and NLO $\supset$ magnetic

- General $N^{+}LO$ theories and solutions
- Easier than GR? Analytics, numerics
- Apply 3+1 techniques to $c \to \infty$ expansion